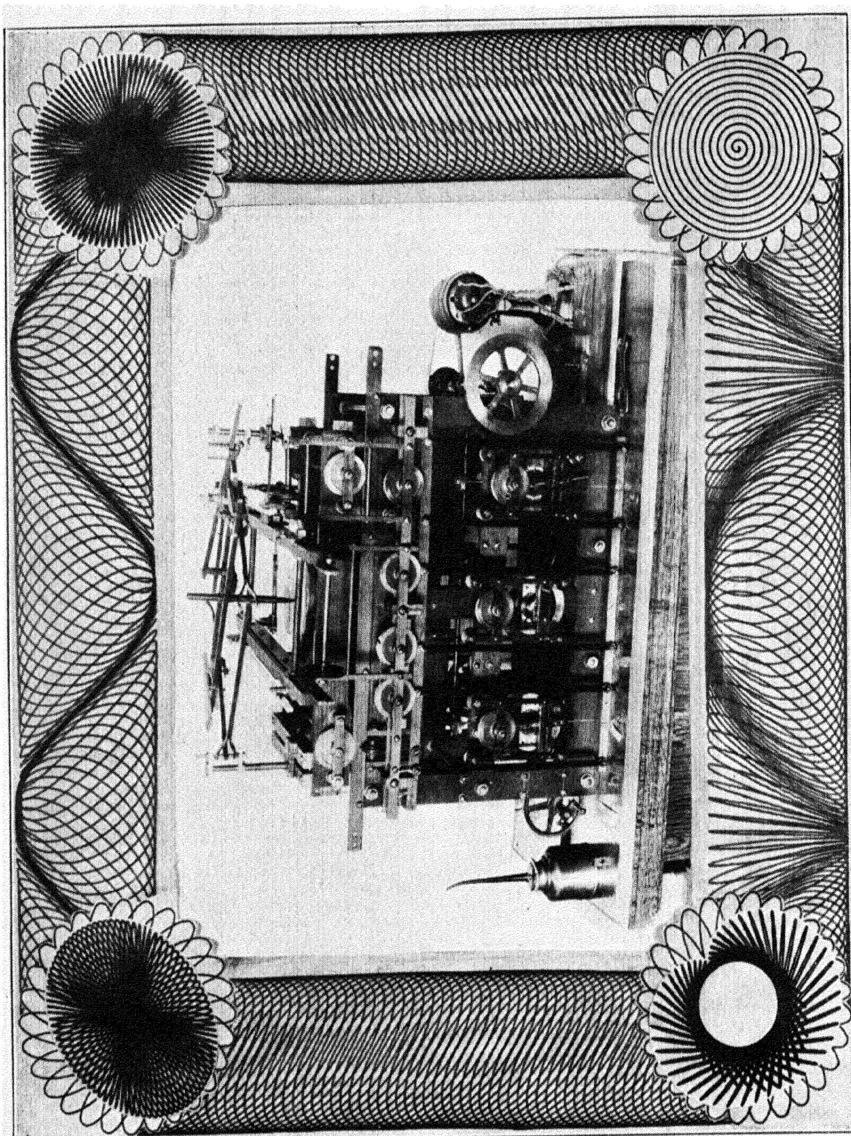

HARMONIC CURVES

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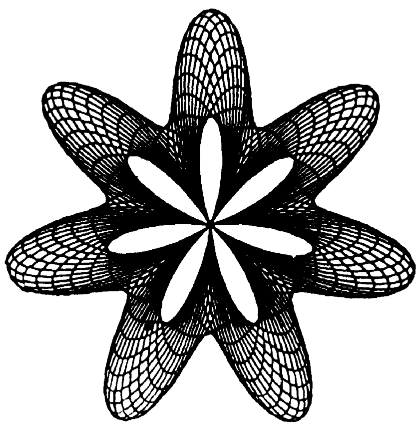

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PREFACE

11. Harmonic curves always captivate the eye by their wonderful beauty and their endless variety. They have that correct proportion in their parts which delights the artist, that simplicity of construction in their apparent complexities which allures the draughtsman, and that fecundity in mathematics which attracts the beginner and holds the veteran.

21. As the chief characteristics just enumerated, their beauty and variety, their ease of construction, and their mathematical richness, are not however the exclusive property of harmonic curves, but may be possessed as well and in varied degrees by other classes of curves, the present book opens with a definition of what the nature of a harmonic curve really is, and how its elements may be varied and compounded.

22. The second chapter discusses the mathematics. As a harmonic curve will be shown to be due to a motion of the tracing point in one or many circles simultaneously, it was necessary to employ trigonometry quite generally. Then in order to make this work available to a larger class of readers, and for this reason not to discourage those whose mathematical acquirements are very modest, it is only in rare cases that mathematics of a higher order than plane trigonometry has been called upon. While on the one hand, the chapter on their mathematics is not really essential to the mechanical construction of harmonic curves, so that it may even be entirely omitted by the draughtsman; on the other, however, to the lovers of mathematics the equations of the curves and the methods of establishing them, together with many other items, may make this the most interesting chapter of the book.

23. The third chapter shows how, in default of a machine, many of the simpler curves may be drawn by plotting, and how the co-ordinates or positions of points may be found.

24. Chapter the fourth describes a few simple, as well as a few complex, machines that have been employed to draw harmonic curves.

25. A fifth chapter is devoted to cycloids, curves generated by a point on a circle as this rolls on another circle. Some novel properties are here presented.

26. The sixth chapter speaks of the beauty of harmonic curves, with some practical suggestions as to how this may be obtained or increased. It also hints at a few of the many agreeable surprises that the drawing of harmonic curves is bound to occasion.

27. Stereoscopic curves are next treated. Ordinary, and even the simplest, harmonic curves may be drawn in such pairs, that when looked at through a stereoscope, or even without such aid, not a plane figure, but a three-dimensional one is seen, and seen so distinctly, that one almost imagines he can run his hands over all its parts.

28. While all the other chapters show how a curve is drawn or put together synthetically, the eighth does the very reverse, and explains how an unknown harmonic curve may be analysed into all its components and their elements, so that when these are found, the identical curve may be drawn on a machine. This matter calls for close study, but except for one fundamental principle and formula taken from the calculus, the rest of the mathematics does not transcend trigonometry.

29. The ninth chapter takes the straight line, the circle, and the ellipse, and shows how these may be made variable in a harmonic way, what figures they create when they rotate, or move along in a straight line or in a circle.

And finally, a tenth chapter or an Appendix includes a reprint of four technical articles by the author.

31. The paragraphs and figures in this book have been numbered on the decimal system which is now becoming more and more universal. A hundred numbers have been assigned to a chapter, just as they are to a block in a city. The full hundred is never used in either case. But as the first digit represents the chapter or the block, such as the 9 in 924, this device quickly points out the approximate locality, and presents a convenience which the old continuous system does not possess. The two other digits, such as the 2 and 4 in 924, then rapidly lead one to the exact spot. While zeros or cyphers are used in many books, such as the 0 in 230 or 501, they are excluded from the present one, so that the first paragraph is numbered 111.

32. The diagrams and figures are given the numbers of the paragraphs in which they occur or in which they are explained. Thus, if in looking over the List of Illustrations one is interested to know what a variable and rotating ellipse might look like, and sees that it is numbered 943, he need but turn to paragraph 943 (in the 9th chapter, of course), to find both the figure and its explanation. The paragraph number thus serves as a page number, and by replacing it, simplifies

matters by using only one number instead of the three for the paragraph, the figure, and the page.

33. But as no system is perfect, the difficulty arises as to what is to be done when a paragraph has several figures. The solution here used is to adhere to the rule just laid down, of giving the figures the number of their paragraph, and to annex to them the letters of the alphabet, *A. B. C.* . . . Thus figure 545*A* is the curve marked *A* in paragraph 545. Letters were preferred to decimals for this subdivision for the sake of greater clarity, thus always employing three digits and only three, except in this Preface, where two are used. As the lettered figures are generally in a group, the finding is so easy and so obvious that the foregoing remarks are practically redundant. Another reason was that when letters are used, 26 of them may be employed, while with numbers there would be only 9 available, paragraph 625 actually having 11 of them from *A to K*.

34. An exception had to be made in the ribbon figures of Chapter IX. On account of their length these had to be grouped in order to save space. But in no case is a figure too far from its official position, except, of course, when it is on the frontispiece or is referred to in two or more places.

41. The author hesitated for quite a while before beginning to write this book, much as he was inclined to do it. The first reason that deterred him was that the subject of harmonic motion is thousands of years old, since it had figured so strongly in the Ptolemaic conception of our planetary system. It is in use even today in predicting the positions of the planets, although the ancient geometric form has given way to the modern algebraic one of a FOURIER series.

42. A second deterrent factor was that the author did not consider himself sufficiently acquainted with the literature of mathematics, to know what had already been written on this subject.

43. On the other hand, the fact that the *American Mathematical Monthly* had published the four articles which he had submitted to it, and which are herewith reprinted in the Appendix through the courtesy of the editors; along with the hundreds of fine curves which his machine had drawn and which he had never seen anywhere in print, finally induced the writer to ask the advice of John G. Hagen, S. J., Director of the Vatican Observatory and author of the "Synopsis der Höheren Mathematik." His answer was as clear as it was firm: "By all means write that book. There is no literature on that subject.

Explain your machine and illustrate it, and add a chapter on all the machines you can find."

44. Francis J. Gerst, S. J., of St. Louis University, who had lately won his Ph. D. degree in mathematics at Johns Hopkins University and had made a personal survey of the mathematical library of this institution and of that at Ann Arbor, was equally emphatic in his decision: "As far as I was able to ascertain, there is no literature on that subject, only casual references in various books."

45. Encouraged by this advice, the writer began his work. As it advanced, one chapter after another sprang into being, the curves were put in order, new ones were drawn for definite purposes, and the equation of every one of them came within reach. The CREIGHTON machine was presented before the Mathematics Section of the Nebraska Academy of Science in May, 1924, and the author was urged to accelerate the publication of the book.

51. The object of this book is to present the subject of harmonic curves in its general outlines and in a somewhat popular form. Just enough of mathematics has been used to interest those that are fond of it, and to entice them to further and original study. There was no intention to treat the matter exhaustively, much less to hint at the multifarious applications of every species of harmonic, and especially of cycloidal, apparatus in machinery. In the sense in which it was written, this book may be called a pioneer work with the avowed object of encouraging the study of harmonic curves.

CHAPTER I

SIMPLE AND COMPOUND HARMONIC MOTION

Simple harmonic motion may be defined both from a geometric and from an algebraic point of view. The former is the easier and will be treated first.

The Geometric Definition

111. Simple harmonic motion, geometrically defined, is uniform circular motion projected upon a straight line, which is generally a diameter of that circle. As projections upon parallel lines and planes are identical, the illustration of a revolving wheel, Fig. 111, turned edgewise

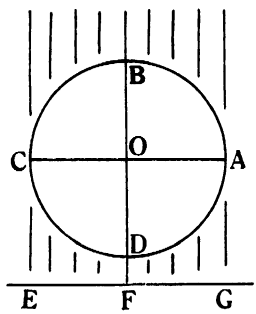

FIG. 111. Simple Harmonic Motion geometrically defined.

to the sun, is a very apt one. The shadow *EG* of such a wheel is then a straight line. When a point *A* on its rim, such as a crank pin, is taken, its shadow will fall upon *G*. As the wheel revolves in an anti-clockwise direction, the crank pin starts to move upward with the full velocity of rotation. But this lasts only for an instant, because it at once begins to lose in upward speed and to develop a lateral motion towards *BD*, so that as the shadow on the line *EG* can show only this lateral motion, it begins to move away from *G* towards the left. Then as the crank approaches *B*, its shadow moves away from *G* towards *F* with increasing speed, so that when the pin is at *B* and has for the moment its entire circular velocity equivalent to a horizontal one, the shadow passes *F* with this same speed. Then the movements just

described repeat themselves in the inverse order. The shadow leaves F with decreasing speed, and when the pin is at C, the shadow is momentarily at rest at E. Then the shadow moves from E towards F and G in exactly the same way as it moved from G to E. When the pin arrives at A and its shadow at G, the cycle is complete. Continued rotations of the wheel can then do nothing more than merely repeat the cyclic motion of both pin and shadow.

112. Here the uniform circular rotation of a point in the circle $ABCDA$ is transformed into the variable, rectilinear, oscillating one $GFEFG$. This motion of the shadow in the line EG is called simple harmonic. While for the sake of a physical illustration the line EFG was taken outside of the circle, the simple harmonic motion along EFG may be transferred perfectly in all its details to the diameter COA, when the more mathematical idea of projection, orthographic projection, is employed, by means of which points on a curve are transferred, or projected, upon a straight line by a system of parallel lines.

113. Mathematically it would not be correct to say that the motion of the crank pin appears to be simple harmonic when the wheel is viewed edgewise, because then, while there is a real projection, this projection is not orthogonal, because the lines of sight are not parallel. Physically there is no noticeable difference when the point of view is very far away, and none whatever when it is at infinity, an expression that is frequently used.

114. Simple harmonic motion may thus be defined to be the *lateral* motion of a point rotating with uniform speed in a circle, or its *vertical* motion, or its motion parallel to the X and Y axes OA and OB, or for that matter parallel to any diameter or line in its plane.

This simple harmonic motion is very easily produced mechanically with all desirable precision. A crank pin is inserted in a slotted bar which is constrained to slide between fixed guides at right angles to the slot. Illustrations of this are to be seen in chapter IV on Machines.

115. Fig. 115 shows simple harmonic motion more in detail. Here the vertical motion is studied, while it was the horizontal one in Fig. 111. The point starts at 0° and rotates anti-clockwise, from 3 o'clock on the dial backwards through 2, 1, 12, 11, and so on, in the conventional mathematical way. The height of the point above the horizontal diameter at any time is equal to the sine of the angle that its radius makes with this diameter, the radius being taken as the unit of measurement. These heights, plus and minus, are marked on the straight

line to the right of the circle in the figure, on which the point would be projected if the rays of the sun came horizontally from the left.

When the generating point revolves with uniform angular speed in its circle, its positions correspond to equal time intervals. Its pro-

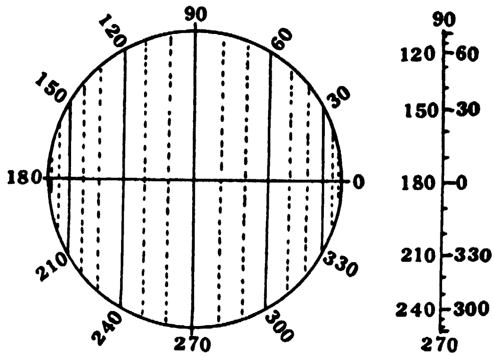

FIG. 115. Simple Harmonic Motion illustrated.

jections then likewise represent equal time intervals. They show that the motion is most rapid in the middle of the path, then diminishes in speed near the end, where it is momentarily zero, after which its motions and positions are repeated in the inverse order until it passes the middle with its maximum speed, only to reduce and reverse it again as before.

116. The geometric study of simple harmonic motion ignores the speed of the point and leaves that to algebra, but it emphasizes its *position* instead. As position is more readily understood than speed, the geometric method is more easily grasped by the ordinary student. It is also the more convenient one in practice, because speed, and even its uniformity, is no essential element in a wheel machine. This may therefore be run with a variable speed, it may be stopped at any point, and even reversed.

117. An analysis of simple harmonic motion discloses three elements, the period, the amplitude, and the phase. The period is the time required for one complete cycle, from the moment the particle passes through any point until it passes through the same point again in the same direction. While, as just said, time is no essential factor in the geometric method, *comparative* time is. In wheel machines the periods are inversely as the number of the cogs on the wheels, or as their radii or diameters.

The amplitude is the radius of the circle, or in the projection it is the distance from the middle of the line to its ends, so that the whole line is double the amplitude.

The phase is the fraction of the cycle that has elapsed since the point last passed through its mid-position going in the positive direction. On the circle Fig. 115 it would be the angular position of the point measured from 0° towards 360°. The phase may thus be neatly indicated on a dial in wheel machines, and read to single degrees or even to fractions of degrees. As merely the up-and-down or right-and-left motions of a crank pin are called for, it makes no difference in practice whether the crank starts at 0° in Fig. 115 and revolves anti-clockwise in the usual mathematical way, or starts at 180° and revolves clockwise. For the sake of mathematical consistency and simplicity, however, the customary direction shown in Fig. 115 will be adhered to.

The Algebraic Definition

121. Simple harmonic motion, defined algebraically, is the motion of a particle which is urged towards a fixed point with a force directly proportional to its distance from it. At first thought it seems impossible that the force should be twice as great when the particle is twice as far away, since other forces, like those of gravity and magnetism, diminish with the distance, and even with the square of the distance. This is, however, perfectly true, and is the case in a musical string or wire. As this must be an elastic body, it obeys Hooke's law, that the farther its midpoint is drawn aside, or the more the wire is stretched, the greater will be the force that will try to restore the wire into its position of rest as a straight line. But when, after being released, it flies back into this position, it is urged all along its path with a force which, it is true, is constantly diminishing, but is such that the sum total of all these urgings makes it overshoot the desired position with its maximum speed. It is then pulled back with an ever increasing force, and brought to the position of rest, only to be urged back and made to repeat its former motion. It thus oscillates to and fro with a motion that will presently be proved to be simple harmonic. And it is because elastic bodies that vibrate in this way give forth a musical sound, that such motion gave rise to the word harmonic, and simple, because at the present stage there is question of a first analysis. A musical string, however, that produces its fundamental note only, does really vibrate with simple harmonic motion. That this is mathematically correct will now be proved by projecting this motion upon a circle which has for its diameter the extent of the excursions, that

is, the double amplitude. This algebraic process is thus the reverse of the geometric one. It is as if the sun cast the shadow of the line on the circle.

122. Let the path be COA in Fig. 122, and let the particle be at H, and $OA = a$, $OH = d$, and $m =$ mass of the particle. Then the force at A is mak and at H mkd, k being a factor showing that the forces are proportional to the distances. The average or mean force acting through the distance AH is then $\frac{1}{2}\, mk\ (a + d)$. As the distance moved over is $AH = a - d$, the energy expended is equal to the average force times the distance, or

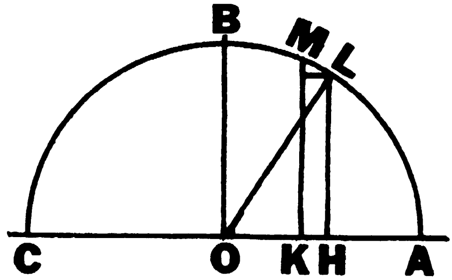

FIG. 122. Simple Harmonic Motion algebraically defined.

$$\tfrac{1}{2}\, mk\ (a + d)\ (a - d) = \tfrac{1}{2}\, mk\ (a^2 - d^2).$$

Mechanics has another expression for this kinetic energy, $\frac{1}{2}mv^2$, in which v is the velocity, so that

$$\tfrac{1}{2}\, mv^2 = \tfrac{1}{2}\, mk\ (a^2 - d^2)$$

or

$$v^2 = k\ (a^2 - d^2).$$

To transfer this velocity to the circle, let the distance HK represent what the particle would move over with the speed v in a unit of time. When $v = HK$ is very small, it is equal to the arc s or ML times the sine of the angle LMK or its equal LOA or θ, that is, $v = s \sin \theta$.

Now, as $\sin^2 \theta = 1 - \cos^2 \theta = 1 - \dfrac{d^2}{a^2} = \dfrac{a^2 - d^2}{a^2}$, it follows that

$$v^2 = s^2 \sin^2 \theta = \frac{s^2\ (a^2 - d^2)}{a^2}.$$

Putting the two values of v^2 equal to one another,

$$\frac{s^2}{a^2}\ (a^2 - d^2) = k\ (a^2 - d^2), \text{ so that } k = \frac{s^2}{a^2}, \text{ or } s^2 = a^2 k.$$

As this expression does not contain *d*, the velocity *s* of the particle, when projected from the straight line *COA* upon the circle *CBA*, is constant. Since the period of the motion of the particle along the line *COA* is evidently the same as that of its projection on the circle, the "algebraic" motion of the particle along the line *AOC*, according to the law that it is urged towards *O* with a force proportional to its distance from *O*, is projected upon the uniform "geometric" motion of its shadow in the circle *ABC*, and is therefore simple harmonic.* The algebraic and geometric definitions of simple harmonic motion are thus not only in perfect accord, but are reciprocals of each other, as each takes as its starting point the conclusion of the other.

123. As the pendulum is used in a certain class of harmonic motion machines instead of a wheel, it will be necessary to show that its vibrations are simple harmonic. This is done in the algebraic way.

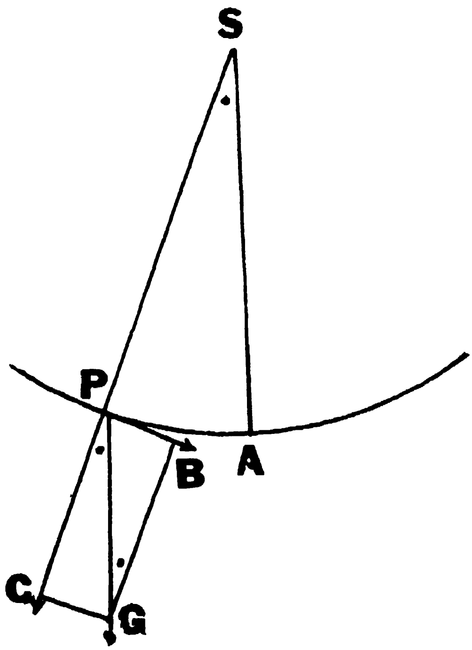

FIG. 123. The Pendulum moves with Simple Harmonic Motion.

A simple pendulum in physics is defined to be a single material particle hung by a string without weight from a fixed support and subject to the force of gravity alone. Such a pendulum is, of course,

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* Taken mostly from "Mechanics, Molecular Physics and Heat," by R. A. Millikan Ginn and Co. 1903. pp. 88, 89.

an ideal one. It may be approximated in reality by making the supporting string or wire or rod very light and the weight or bob very heavy. But even this is not absolutely necessary.

When the bob or particle P in Fig. 123 has been drawn aside from its position of rest at A, the force of gravity pulls it in the direction PG. Let the length of this line PG represent the force of the pull. It may be decomposed into the force PC which pulls the string, and the force PB which urges the particle down the arc PA. Now $PB = PG \sin\theta$. When θ is small,

$$\sin\theta = \theta = \frac{PA}{PS}, \text{ so that } PB = PG\frac{PA}{PS}.$$

As the weight of the bob PG and the length of the string PS are constant, PB varies as PA, that is, the force which urges the pendulum towards A varies directly as the distance PA. The motion of the pendulum is therefore simple harmonic.

124. It is also isochronous, that is to say, its period or cycle is constant, even when the extent of its vibrations or its amplitude diminishes, it being always understood that the angle θ is so small that its arc is practically equal to its sine.

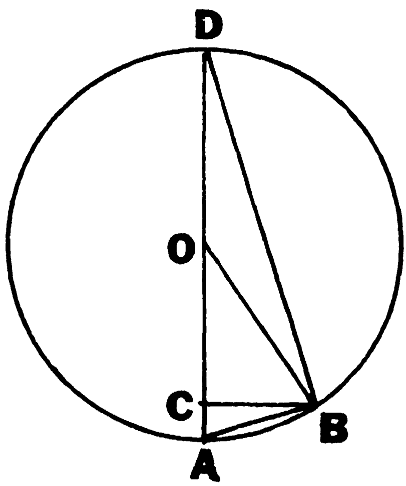

FIG. 124. The Isochronism of the Pendulum.

It is evident that when the pendulum passes through its lowest point A in Fig. 123, it does so with its maximum velocity, because it is being continually urged towards it while it approaches it, and being continually held back while it leaves it. It is further evident from the

preceding Figs. 111, 115, 122, that this momentary maximum velocity is equal to the uniform rotary velocity of the point in the circle, $a\omega$, when $a =$ radius,* and ω angular velocity. Since $\omega = 2\pi/t$, t being the time required to complete one revolution or cycle, the maximum velocity is $a\omega = 2\pi a/t = v$. The kinetic energy is then $\frac{1}{2}mv^2 = m2^2\pi^2a^2/2t^2 = 2\pi^2a^2m/t^2$. This kinetic energy at the lowest point A in Fig. 124 must be equal to the potential energy at its highest point B. This last is mgh, where $h = AC =$ height. Therefore $2\pi^2a^2m/t^2 = mgh$. In order to eliminate h, let a circle be drawn with $OA = l$ as radius about the point of suspension O. Then since the small triangle ACB is similar to the large one ABD,

$$\frac{AC}{AB} = \frac{AB}{AD} \text{ or } AC = h = \frac{\overline{AB}^2}{AD} = \frac{a^2}{2l} \text{ when the arc is small.}$$

Therefore $\dfrac{2\pi^2a^2m}{t^2} = mg\dfrac{a^2}{2l}$ or $\dfrac{4\pi^2}{t^2} = \dfrac{g}{l}$, and $t = 2\pi\sqrt{\dfrac{l}{g}}$ the usual formula for the pendulum, in which t is the time of one complete period or double swing, l the length of the pendulum, and g the acceleration of gravity = 980 cm = 32 feet approximately. For a semi-vibration or a single swing from one side to the other but not back again, as it is generally taken for a pendulum, the 2 must be cancelled from the formula. As this formula contains only constants, and especially as a, the amplitude of the swing, is not in it, nor the mass or weight m, the vibrations must remain isochronous even when the amplitude diminishes. The period of a pendulum is therefore invariable, as long as its length is the same.†

125. As the time of vibration of a rod pendulum of convenient length is rather short, it may be lengthened by fastening a lighter weight above the point of suspension (421). This will then interfere with the bob, rising when the bob falls, and vice versa. It practically reduces the value of g, and thus increases t.

Graphic Presentation

131. When a pen or a stylus, controlled by a pendulum or a wheel machine, moves with simple harmonic motion, its path would ever

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* That is, the amplitude, as defined before, not AS in Fig. 123, nor OA in Fig. 124.

† This proof was taken from Watson's Physics, pp. 129, 130. Longmans, Green and Co. 1900.

remain the same to no purpose unless either the pen itself or the paper under it were given a motion at an angle, generally at a right angle, to it. This may be done in three ways. The paper may be moved 1. with uniform linear speed, 2. with simple harmonic motion, or 3. with a uniform rotary velocity. All these will be studied in detail in the following two chapters.

Compound Harmonic Motion

141. All along in this Chapter the pen has been supposed to move with simple harmonic motion. It may also be given a compound harmonic movement. This may be done in various ways. The first method is to add a second, a third, or more components, each with its own period, amplitude, and initial phase, exactly like the little waves and wavelets that may be seen on large waves. It is in this way that musical strings and wires generally vibrate. Besides the fundamental note, they give out also what are called harmonics, that is, musical sounds with 2, 3, 4, etc., times as many vibrations, each of which when alone would make the particles of the wire vibrate with simple harmonic motion, but with varying amplitudes or degrees of loudness, thereby giving the characteristic called timber which distinguishes the same note as given by various instruments. The particles of the wire then move at any moment with the algebraic sum of the components in their respective phases. It is as if a series of pendulums could be hung on one another without interference, all vibrating in the same plane, the bob of each serving as the point of suspension of the next, the lowest bob then carrying the pen. While this last conception cannot be made a reality in the manner it has been stated, it illustrates the way that pendulums may be compounded in principle. In practice two independent pendulums, vibrating in the same plane, but on opposite sides of a table, may be employed, one moving the pen and the other the paper or the plate. In wheel machines there is no limit to the number of such components in series.† They are called harmonic synthetizers, examples and illustrations of which may be found in chapter iv.

142. A second way of producing compound harmonic motion is to use two simple harmonic motions at right angles, one for the pen and the other for the paper, or both for the pen. This method might also be classed under simple harmonic motion, as has been done before (131). The movement may then be more compounded by having two

† Miller (849) has used as many as 30, and the Tide-Predicting Machine of the United States Coast and Geodetic Survey employs 36. (*Pop. Astr.*, xx, 269)

or more components on the X axis and two or more also on the Y axis, each of course with its own period, amplitude, and initial phase.

143. A third and more complex way is to put the one, two, or more components in both X and Y directly on the pen, and then to pull the paper along with uniform rectilinear speed or to rotate it on a disk. With wheel machines the compounding may be extended indefinitely. With pendulums this seems to be limited to four of them. A rotating disk under a pendulum appears never to have been used with success. The chief difficulty is probably the uniformity or rather the exactitude of the period of its rotation, because time is an essential element to a pendulum machine.

Summary of Chapter I

(111) Simple Harmonic Motion, geometrically defined, is uniform circular motion projected upon a straight line. (114) It is the vertical or horizontal motion of a crank pin. The *motion* of a point (116) is ignored in the geometric method, and its *position* used instead. The elements of simple harmonic motion (117) are the period, the amplitude, and the phase.

(121) Simple Harmonic Motion, algebraically defined, is the motion of a particle which is urged towards a fixed point with a force directly proportional to its distance from it. A pendulum (123) swings with simple harmonic motion, and its vibrations (124) are isochronous, or its period constant.

(131) The simple harmonic motion of a crank pin or a pendulum may be recorded graphically on paper by moving this 1. with uniform linear speed, 2. with simple harmonic motion, or 3. with a uniform rotary velocity.

(141) Compound Harmonic Motion is the combination of two or more simple harmonic movements, each with its own period, amplitude, and initial phase. These components may be connected in series or at any angle. The most complex combination (143) is obtained by putting one or more components in both the X and Y axes and moving the paper with uniform rectilinear or rotary speed.

CHAPTER II

THE MATHEMATICS OF HARMONIC CURVES

The mathematics of harmonic curves, as presented in this chapter, will, as a rule, not transcend plane trigonometry, although elementary ideas will be borrowed from analytical geometry and the calculus.

I. Sine Curves

211. In the geometric definition of simple harmonic motion it was said (115) that the distance of the pen from its middle position at any moment was the amplitude times the sine of the phase. Thus in Fig.

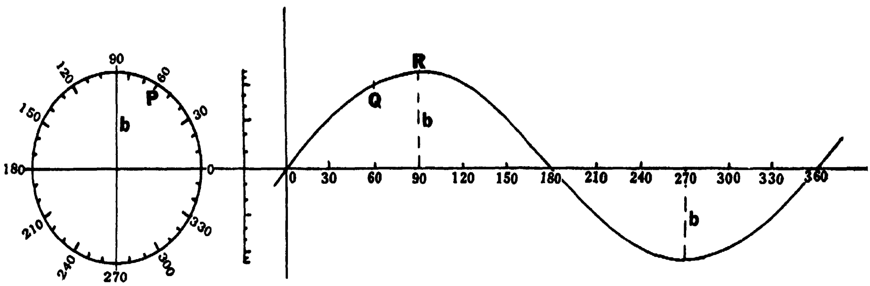

FIG. 211. A Simple Sine Curve, $y = \sin x$.

211, if b is the amplitude or the radius of the circle, and the harmonic motion is vertical, the ordinates of points, or their heights above the X axis, are $b \sin \theta$, where θ is the phase. Due regard must be had, of course, to algebraic signs.

The wavy curve is called a sine curve. Horizontal distances or abscissae, are θ, the arc in a unit circle, so that the length from $0°$ to $360°$ is $2\pi = 2 \times 3.1416$, or $2\pi a$, if another scale is used. Thus if the point P in the circle is taken at $60°$, it is transferred to Q on the curve, the co-ordinates of Q being then

$$x = a\theta = \text{arc of } 60° = 0° \, P \text{ on a circle with radius } a,^*$$
$$y = b \sin \theta = b \sin 60°.$$

When a and b are unity, the parametric equation, that is, the equation which gives the values of x and y separately, is then

$$x = \theta.$$
$$y = \sin \theta.$$

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* In the figure $a = b$.

When these two equations can be combined into one by the elimination of θ, a single simpler and more usual and desirable equation results, which is then called a Cartesian equation. On the other hand, by retaining θ in polar curves (231) and eliminating x and y by putting $\rho^2 = x^2 + y^2$, the equation is called a polar equation. This transformation of parametric, Cartesian, and polar equations into one another is not always possible, or when it is possible, the conversion may be practically too cumbersome. In the present instance the Cartesian equation, $y = \sin x$, is, of course, very readily obtained. It is an algebraic representation of a pure sine curve, called the sinusoid. It is drawn mechanically by moving the paper with such uniform speed to the left* that in one cycle of the pen it advances the distance of $2\pi = 2 \times 3.1416$.

This simple equation may be modified in three ways, according to the three elements of simple harmonic motion, any one or two or all of which may be changed.

212. *The Period.* The equation for Fig. 212 is $y = \sin 16x$. For example, by taking x or $\theta = 5°37'.5$, $y = \sin 16 \times 5°37'.5 = \sin 90° = 1$.

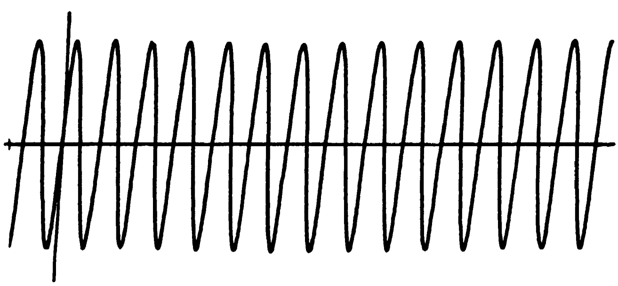

FIG. 212. A Sine Curve, $y = \sin 16x$.

The amplitude is the same as in Fig. 211, but the period is shortened. There are 16 waves in Fig. 212 to one in Fig. 211, the pen now running through a cycle 16 times as fast.†

Instead of saying that the pen runs through its cycle 16 times as fast while the paper moves with its usual speed, it is more in accord with the fact to say that the pen completes its cycle in the usual time while the paper advances with 1/16 of its former speed. The parametric equation is then

$$x = \tfrac{1}{16}\theta = g\theta, \qquad y = \sin \theta.$$

* By means of a roller with a unit radius (466).

† In wheel machines the periods are proportional to the reciprocals of the radii or the number of their cogs.

By combining these two equations, the former one of

$$y = \sin 16x \qquad \text{or} \qquad y = \sin (\theta/g)$$

is, of course, obtained, so that the mathematics of the two concepts is the same. But, as will be seen later (216), the second method gives the velocity g of the paper, which it is necessary to know in Rectangular-Sine Curves. In the present case $g = 2\pi/16$, that is, 1/16 of the circumference of the unit circle $= 0.393$, which may be measured directly on the paper, since it is the length of one wave or one cycle of the pen.

213. *The Amplitude.* The smaller curve in Fig. 213 has the equation

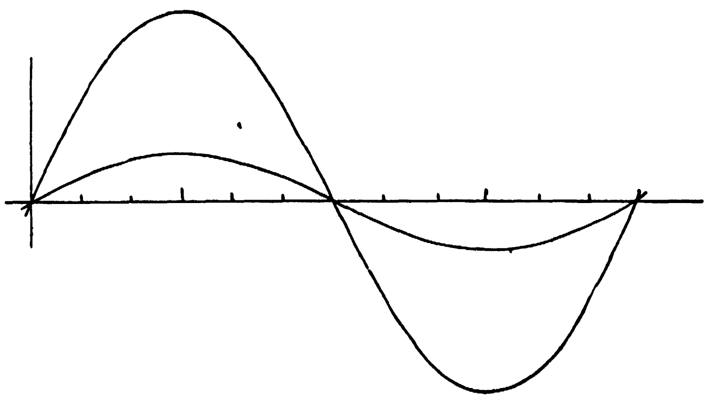

FIG. 213. Two Sine Curves, $y = \frac{1}{2} \sin x$ and $y = 2 \sin x$.

$y = \frac{1}{2} \sin x = b \sin \theta$. The amplitude and all the ordinates are one half of what they are in Fig. 211, while the period is the same. Instead of diminishing the amplitude, it may also be increased, as is the case with the larger curve in Fig. 213, the equation of which is $y = 2 \sin x$. All its ordinates are double the corresponding ones in Fig. 211.

214. *The Initial Phase.* Suppose that in Fig. 211, the Y axis were to pass through Q. Then, as the abscissa x is always measured from the Y axis, it would not be correct to say that for the point R $x = \theta = 90°$. It is really only the arc of 30°. It would be equally wrong to say that $y = \sin 30°$, when it is known to be equal to sin 90°. As the curve crosses the Y axis, where x is always 0, in phase 60°, this phase is an initial one, it is like an index error, and must be added to θ or x when the value of y is to be found, so that the equation would then be

$$y = \sin (x + 60°). \text{ Then, when } x = \theta = 30°,$$
$$y = \sin (30° + 60°) = \sin 90°.$$

215. These three modifications may be written in one general equation

$$y = b \sin (n\theta + \eta)$$

in which b = amplitude, n = periods, η = initial phase. It is still, of course, a sine curve, but not a pure one like $y = \sin x$ in which b and n are unity and η is zero.

216. Compounding is now easy.

$$x = \theta$$
$$y = b \sin (n\theta + \eta) + d \sin (q\theta + \nu) + f \sin (w\theta + \lambda) + \ldots$$

Fig. 926A (in the ninth chapter) is an instance of a compound sine curve with two components. The amplitudes are equal and will be taken as the unit of measurement. The initial phases are 0°. and the periods n and q are in the ratio of 16: 15, so that the equation is

$$x = g\theta$$
$$y = \sin 16\theta + \sin 15\theta,$$

or rather for the sake of clarifying fundamental ideas

$$x = g\theta$$
$$y = \sin \theta + \sin {}^{15}\!/_{16}\theta.$$

This latter shape shows that the curve may be conceived to be a single sine curve $y = \sin \theta$, in which the ordinates are increased by $\sin {}^{15}\!/_{16}\theta$. It is necessary to premise also that the origin, where θ and x and y are zero, is in the center of a lobe, and that θ runs through a cycle of 360° in each of the 16 wavelets, so that when the compound cycle is complete, from the middle of one lobe to the middle of the next, it runs through $16 \times 360°$ for one component and through $15 \times 360°$ for the other.

As θ starts with the value 0° and grows to 90° in the first wavelet, y becomes nearly equal to $+2$, the more nearly equal to it as the ratio of the periods is closer. When $\theta = 270°$, y is nearly equal to -2. The longest double ordinate in the figure—and mathematically this is true of the envelope*—is therefore equal to 4. One fourth of it is then the

* Graphically an envelope is a simple curve which bounds, outlines, silhouettes, or "envelopes" a compound one. Mathematically it is a curve tangent to all the curves of a series which have either one, or two or more mutually dependent, variable parameters, or arbitrary constants, in their equation. Here the variable parameter is $\sin {}^{15}\!/_{16}\theta$. The mathematical definition is, of course, true in every instance, but the graphic one is not, as may be seen in Figs. 959A and B on the frontispiece. Figs. 926B, 927, 932, 935, 936, 958, and many others, however, show a perfect agreement between the two definitions.

unit of the scale. With this unit it is seen that the paper advances the distance 3.75 in one compound cycle of the pen. As there are 16 wavelets in this distance, the paper moves $3.75/16 = 0.235 = g$ in one simple cycle of y. The equation of the curve 926A is then*

$$x = 0.235\,\theta$$
$$y = \sin\theta + \sin {}^{15}\!/_{16}\theta.$$

A more convenient shape will now be to take the unit angle $1/16$ as large, so that

$$x = 3.75\theta$$
$$y = \sin 16\theta + \sin 15\theta.$$

In this shape the value of g is the length of a complex cycle measured in terms of the radius of the unit circle, or in other words, the length of a horizontal section of the figure divided by four times its greatest vertical dimension.

The intention in drawing the figure 926A had been to make its length equal to its height in order to obtain a beaded curve consisting of a series of balls or circles tangent to one another. This object was not attained perfectly (3.75 and 4.00) owing to mechanical impossibilities or inconveniences. And even if it had, the envelope could not have been circular, especially at the junctions of the sections, it is really a double sine curve, so that at these junctions it must have two tangents.

217. The equation given, $x = 3.75\theta$

$$y = \sin 16\theta + \sin 15\theta,$$

may be taken out of its parametric form and put into the Cartesian (211) by substituting the value of θ in y, so that

$$y = \sin {}^{16}\!/_{3.75}\, x + \sin {}^{15}\!/_{3.75}\, x.\dagger$$

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* The coefficient of θ in the second term of y must be a proper fraction, $15/16$ here, and not an improper one, $16/15$. The reason is that as the first term is taken as the fundamental curve, it must have its full number of cycles, 16. In each cycle the increment of its ordinates is the sine of 15/16 of its phases, or an additional $1/16$ less, so that in the 16 cycles the total increase amounts to $16 \times 1/16$ or one whole cycle less, that is, $16 - 1 = 15$ cycles for the second term. This subtractive cycle is not directly visible, because it is implicitly contained in the first term. Now, if the fraction were $16/15$ and the first term had 15 cycles, the increment would be an additional $1/15$ more, or one whole cycle more, $15 + 1 = 16$ for the second term. This additional 16th cycle must then appear in the figure. In both cases therefore the number of the apparent cycles or wavelets is 16, or the greater number, the understanding being that the faster component has only one cycle more than the slower in the complex curve. And this is confirmed by experiment.

† It may be of interest to indicate how the phases of the components may

218. In Fig. 247B will be shown a sine curve with four components and with the axis of X bent into a circle.

II. Rectangular Curves

221. In rectangular curves the pen is given a harmonic motion parallel to both axes. In practice the pen often moves parallel to the Y axis only, and the paper parallel to the X axis. In this case the relative or effective motion of the pen is in a direction opposite to that of the paper.

The parametric equation of a rectangular curve is in a sense a duplicate of that of a sine curve (216).

$$x = a \sin (m\theta + \xi) + c \sin (p\theta + \mu) + e \sin (v\theta + \varkappa) + \dots$$
$$y = b \sin (n\theta + \eta) + d \sin (q\theta + \nu) + f \sin (w\theta + \lambda) + \dots$$

222. It is preferable to write the formula as given, and to express x also as a sine rather than as a cosine, as is commonly done. First, because in Lissajou curves, that is, rectangular curves with only one component in each axis (432), when the initial phases ξ and η are 0°,

$$x = a \sin m\theta$$
$$y = b \sin n\theta$$

x and y are both zero when θ is zero, and the curve passes through the origin. This would not be the case when x is written as a cosine.

Secondly, by omitting ξ and η from the equations, these are much less cumbersome. They are by no means inaccurate, because ξ and η may be added to $m\theta$ and $n\theta$ at any stage of the investigations. They are seldom combined with $m\theta$ and $n\theta$ except when they are equal to 90°. It is only when this happens that it is advisable to write $\cos m\theta$ and $\cos n\theta$.

Thirdly, the sine notation is directly adaptable to dials on wheel machines.

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be found at the points where $y = 0$ and where it is a maximum and minimum, that is, graphically, on the X axis and at the peaks of the curve. Let these phases be called A and B. When $\theta = 0°$, $y = 0$, of course. The first time that y is again zero happens when $A = 180° + \varphi$ and $B = 180° - \varphi$. From this it follows at once that $A + B = 360°$, and

$$A = \frac{16}{16+15} \text{ of } 360° = 185°48'23''.22 \text{ and } B = \frac{15}{16+15} \text{ of } 360° = 174°11'36''.78$$

The second time that $y = 0$, the pases are 2A and 2B, the third time they are 3A and 3B, and so on. The maxima and minima of y then occur midway between these.

Special Cases

223. *First,* the periods are equal, $m = n$. Then

$$x = a \sin(\theta + \xi)$$
$$y = b \sin(\theta + \eta)$$

1) When $\xi = \eta$, $y = \dfrac{b}{a}x$, and the curve is a *straight line* passing through the origin and making an angle φ with the X axis, such that $\tan\varphi = b/a$.

2) When $\eta = 0°$
$$x = a \sin(\theta + \xi)$$
$$y = a \sin\theta$$
and the curve is an *ellipse.**

3) When
$$a = b,\ \eta = 0°,\ \xi = \pm 90°,$$
$$x = \pm a \cos\theta$$
$$y = a \sin\theta$$

and then the Cartesian equation is $x^2 + y^2 = a^2$, and the polar $\rho = \pm a$, so that the curve is a *circle.* The same is, of course, the case when $\xi = 0°$ and $\eta = \pm 90°$ and in general when there is a difference of 90° in the phases, $\xi - \eta = \pm 90°$.†

* The proof is deduced from analytical geometry. By eliminating θ, the Cartesian equation becomes

$$b^2x^2 + a^2y^2 - 2ab \cos.\ \xi \cdot xy - a^2b^2 \sin^2\xi = 0.$$

The coefficients of the variables then show that the curve is an ellipse. Its semi axes are

$$\frac{\sqrt{2}\,ab \sin\xi}{\sqrt{b^2 + a^2 \pm \sqrt{(b^2 - a^2)^2 + 4a^2b^2\cos^2\xi}}}$$

The double sign $\pm$ means that plus is to be used for the semi major or tranverse axis, and minus for the semi minor or conjugate axis. The major axis is inclined at the angle φ to the X axis, such that

$$\tan 2\varphi = \frac{2ab \cos\xi}{a^2 - b^2}$$

The quantities a and b, it is to be noted, are the amplitudes of the original rectangular equation.

† Let $x = a \sin(\theta + \varkappa \pm 90°) = \pm a \cos(\theta + \varkappa)$
$y = a \sin(\theta + \varkappa)$
then $x^2 + y^2 = a^2$, as before.

The rotation is direct, or anticlockwise, when x leads in phase, and inverse, or clockwise, when x lags in phase.* The zero point, from which the angle θ is counted, depends upon the actual initial values of ξ and η. The annexed table will make this clear. The numbers 0, 90, 180, 270, indicate the zero (or starting) points of θ (i. e. when $\theta = 0°$) and have the conventional mathematical meaning, 0° being 3 o'clock on a dial or east or right, 90° 12 o'clock or north or top, etc. The direction of rotation is then found when θ is given a small positive value (234).

	ξ	η	x	y	Starting Point	Rotation
1	$+90°$	$0°$	$a \cos \theta$	$a \sin \theta$	0°	direct
2	0	$+90$	$a \sin \theta$	$a \cos \theta$	90	inverse
3	-90	0	$-a \cos \theta$	$a \sin \theta$	180	inverse
4	0	-90	$a \sin \theta$	$-a \cos \theta$	270	direct

From this table it is seen that when x is, as is usually the case for a circle, expressed as a cosine and y as a sine, the rotation is direct or inverse according as x and y have like or unlike signs.† The starting point, where $\theta = 0°$, however, is shifted 180° when the signs of x and y are reversed, so that, for example, when $x = +a \cos \theta$ and $y = -a \sin \theta$, the starting point is 180° away from the 180° mentioned in the third case, so that it is at 0°. Much use will be made of this test in cycloids in chapter V. (Compare also 226, 233.)

224. *Secondly,* When $a = b$, $m = l$, $n = 2$.

$$x = \sin (\theta + \xi)$$
$$y = \sin (2\theta + \eta)$$

Let $\eta = 0°$, $\xi = 0°$ or $\pm = 90°$, then

$$x = \sin \theta \text{ or } \pm \cos \theta$$
$$y = \sin 2\,\theta = 2 \sin \theta \cos \theta = \pm\, 2x\sqrt{1 - x^2}$$

or $y^2 = 4x^2(1 - x^2)$. (See Fig. 743B.)

225. When $\xi = 0°$, and $\eta = \pm\, 90°$,

$$x = \sin \theta$$
$$y = \sin (2\theta \pm 90°) = \pm \cos 2\theta = \pm\, (\cos^2\theta - \sin^2\theta)$$
$$= \pm\, (1 - \sin^2\theta - \sin^2\theta) = \pm\, (1 - 2x^2)$$

and $2x^2 = 1 \mp y$, a double *parabola.*

* This is also true for all Lissajou curves. When the periods are unequal, so that x alternately leads and lags, the direction of the pen's rotation changes accordingly.

† The reverse is also true; that when x is a sine and y a cosine, like signs mean an inverse rotation and unlike signs a direct one.

Only the one corresponding to the plus sign before y is shown in Fig. 225. The other one, with the minus sign, is symmetrical to the first in regard to the X axis. It may be seen by turning the page

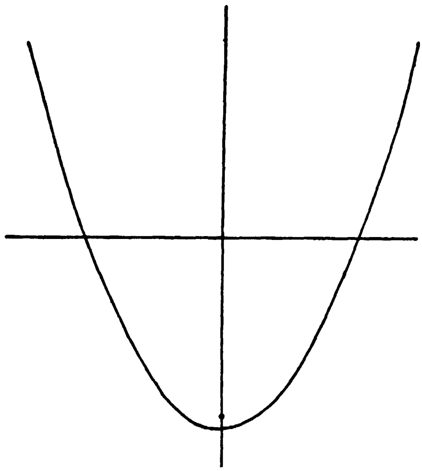

FIG. 225 The Parabola.

halfway round. The same double parabola results when both ξ and η are equal to $\mp 90°$. The pen, of course, draws only one of these parabolas, because ξ and η can have only one value at a time.

226. *Thirdly.* Let the pen move in two circles, the center of one being on the circumference of the other, and let their radii and periods show reciprocity, so that the equation is

$$x = \mp m \cos n\,\theta + n \cos m\,\theta.$$
$$y = m \sin n\,\theta - n \sin m\,\theta.$$

then the curve is a circular cycloid, the upper sign in the first term of x denoting an epicycloid, and the lower one a hypocycloid. The subject of cycloids is treated more fully in Chapter V.

227. Examples of other rectangular curves are given in many places, such as, 713, 717, 743 B and C, 951, 955, which have only one component in each axis. In 741, 743 D and E, there are two in Y to one in X. In 724A, 742, 926B, 934, 944, there are two in each axis.

III. Rectangular-Polar Curves

In this class the pen draws a rectangular curve while the disk is rotating under it. Taking the case of one component only in each axis, the equation of the rectangular curve is (222)

$$x = a \sin m\theta$$
$$y = b \sin n\theta.$$

Here the starting phases ξ and η are omitted, or taken as 0°, in order to simplify the equations. They may be supplied at any time, as will be seen later (233).

231. The origin is now to be transferred from A in Fig. 311 to the point O, whose co-ordinates relative to A are AC and AB or $-\alpha$, $-\beta$, and the axes are to be turned through the angle θ. The formula for this operation, as given in analytical geometry, is

$$x_0 = -\alpha + x \cos\theta - y \sin\theta$$
$$y_0 = -\beta + x \sin\theta + y \cos\theta,$$

in which $x_0 y_0$ are the old x and y, and $x\,y$ the new ones. So that

$$a \sin m\,\theta = -\alpha + x \cos\theta - y \sin\theta$$
$$b \sin n\,\theta = -\beta + x \sin\theta + y \cos\theta.$$

From these it follows that

$$x = (a \sin m\,\theta + \alpha) \cos\theta + (b \sin n\,\theta + \beta) \sin\theta$$
$$y = -(a \sin m\,\theta + \alpha) \sin\theta + (b \sin n\,\theta + \beta) \cos\theta.$$

As the coefficients in the parentheses are the same respectively for the first and second terms, and one multiplies the sine and the other the cosine of the same angle, the pen moves in a circle which has the first parenthesis as a radius, while the center of this first circle moves in a second one which has the second parenthesis as a radius, the center of this second circle being the center of the disk. See the dotted arcs in Fig. 312 in the next chapter, which will be explained better later on.

232. But as the radii of these two circles are variable, the conceived cycloidal motion can be only momentary, and it would be impossible to practically plot the curve. They may be made constant by a trigonometric transformation. In the "Smithsonian Mathematical Formulae and Tables of Elliptic Functions," Washington, D. C., 1923, Formulae No. 3.150 are

1. $\cos nx \cos mx = \frac{1}{2} \cos (n - m)\, x + \frac{1}{2} \cos (n + m)\, x.$
2. $\sin nx \sin mx = \frac{1}{2} \cos (n - m)\, x - \frac{1}{2} \cos (n + m)\, x.$
3. $\cos nx \sin mx = \frac{1}{2} \sin (n + m)\, x - \frac{1}{2} \sin (n - m)\, x.$

These formulae may be established from the ordinary ones for the sines and cosines of the sum and difference of two angles by proper addition and subtraction. By applying them to the present case,

$$\sin\theta \sin n\theta = -\tfrac{1}{2}\cos(n+1)\theta + \tfrac{1}{2}\cos(n-1)\theta$$
$$\cos\theta \sin n\theta = \tfrac{1}{2}\sin(n+1)\theta + \tfrac{1}{2}\sin(n-1)\theta$$
$$\sin m\theta \cos\theta = \tfrac{1}{2}\sin(m+1)\theta + \tfrac{1}{2}\sin(m-1)\theta$$
$$\sin m\theta \sin\theta = \tfrac{1}{2}\cos(m-1)\theta - \tfrac{1}{2}\cos(m+1)\theta$$

By substituting the above values,

$$x = \tfrac{1}{2}a\sin(m+1)\theta + \tfrac{1}{2}a\sin(m-1)\theta + \alpha\cos\theta$$
$$-\tfrac{1}{2}b\cos(n+1)\theta + \tfrac{1}{2}b\cos(n-1)\theta + \beta\sin\theta$$

$$y = \tfrac{1}{2}a\cos(m+1)\theta - \tfrac{1}{2}a\cos(m-1)\theta - \alpha\sin\theta$$
$$+\tfrac{1}{2}b\sin(n+1)\theta + \tfrac{1}{2}b\sin(n-1)\theta + \beta\cos\theta.$$

If $r = \sqrt{\alpha^2 + \beta^2}$ = distance between the first and second origin $= OA$ in Fig. 311, and φ = angle AOX = their relative position angle, so that $\alpha = OB = r\cos\varphi$ and $\beta = AB = r\sin\varphi$, then by replacing φ and β by these values, it is seen that

$$\alpha\cos\theta + \beta\sin\theta = r(\cos\theta\cos\varphi + \sin\theta\sin\varphi)$$
$$= r\cos(\theta - \varphi)$$
$$-(\alpha\sin\theta - \beta\cos\theta) = r(\sin\theta\cos\varphi - \cos\theta\sin\varphi)$$
$$= r\sin(\theta - \varphi).$$

233. The General Equation for a rectangular polar curve with one component to each X and Y, that is, for a Lissajou curve, drawn on a rotating disk, is then

$$x = \tfrac{1}{2}a\sin(m+1)\theta + \tfrac{1}{2}a\sin(m-1)\theta$$
$$-\tfrac{1}{2}b\cos(n+1)\theta + \tfrac{1}{2}b\cos(n-1)\theta + r\cos(\theta-\varphi)$$
$$y = \tfrac{1}{2}a\cos(m+1)\theta - \tfrac{1}{2}a\cos(m-1)\theta$$
$$+\tfrac{1}{2}b\sin(n+1)\theta + \tfrac{1}{2}b\sin(n-1)\theta - r\sin(\theta-\varphi).$$

The starting phases may now be inserted in this wise

$$x = \tfrac{1}{2}a\sin\Big((m+1)\theta + \xi\Big) + \ldots$$
$$\tfrac{1}{2}b\cos\Big((n+1)\theta + \eta\Big) + \ldots$$

It is evident that toting these along all the time would only be carrying useless freight.

In this General Equation the coefficients $\tfrac{1}{2}a$, $\tfrac{1}{2}b$, r, are all constant and, as before, multiply respectively the sines and cosines of the same angles. It may now be said in truth that the pen really does continually move in a circle with radius $\tfrac{1}{2}a$, that the center of this first circle moves in a second circle also with radius $\tfrac{1}{2}a$, the center of

this second moves in a third circle with radius $\frac{1}{2}b$, the center of this third in a fourth also with radius $\frac{1}{2}b$, and the center of this fourth circle in a fifth one with radius r and center at the center of the disk (Fig. 312). The motion of the pen is therefore truly harmonic.*

234. The direction of rotation in these five circles is not, however, the same. In order to find this direction, let $\theta = 0°$, then

$$x = 0 + 0 - \tfrac{1}{2}b + \tfrac{1}{2}b + r\cos\varphi$$
$$y = +\tfrac{1}{2}a - \tfrac{1}{2}a + 0 + 0 + r\sin\varphi.$$

From this it is seen (223) that when $\theta = 0°$

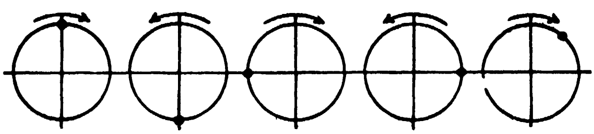

FIG. 234. Zero Points and Directions of Rotation.

the moving point in the first circle is at 90° or on top, in the second it is at 270° or at the bottom, in the third at 180° or at the left, in the fourth at 0° or at the right, and in the fifth at φ degrees.

Now, when θ begins to grow, as the first terms of x and y in the General Equation are both plus, the sine and cosine must be both plus, and the point moves from 90° into the first quadrant. The rotation of the first circle is therefore inverse or clockwise.

In the second terms of x and y, as θ grows, x is positive and y negative, so that the point in the second circle enters the fourth quadrant from 270°, and the rotation is direct or anticlockwise.

The third terms show that x is negative and y positive, and that the point moves from 180° into the second quadrant and the rotation is inverse.

The fourth terms are both plus, the point moves from 0° into the first quadrant and the rotation is direct.

The fifth terms $+r\cos(\theta - \varphi)$ and $-r\sin(\theta - \varphi)$ may be written $+r\cos(\varphi - \theta)$ and $+r\sin(\varphi - \theta)$, both positive. It is then evident that every value of θ will diminish $\varphi - \theta$ and cause an inverse rotation.

The direction of rotation in the five circles is then respectively and alternately, inverse, direct, inverse, direct, inverse. The directions may

* The above proof was taken from P. Potron, S. J., as he elaborated it for the Dechevrens Campylograph (451). Its graphic presentation will be found in the following Chapter.

be remembered from the $+1$ and the -1 in the coefficients of θ which must cause rotations respectively with and against the θ of the disk on which the pen must advance on the whole in an inverse or clockwise direction. (See also 223).

The inverse rotation of this fifth circle, the one with radius r, is of special importance when the mathematics of a rectangular — polar curve is to be investigated. When the rectangular curve is started in the first quadrant, as it is in Fig. 311, the disk must then be turned in an anticlockwise direction. The direction may however be kept clockwise when the curve is started in the third quadrant, so that this may be called the first when the figure is looked at from the opposite side. But when there is question merely of drawing a beautiful curve, the direction of rotation is of no importance.

235. The General Equation (233) may now be extended to any number of components of X and Y in the original rectangular curve. Thus for three in each axis

$$\begin{aligned} x = {} & \tfrac{1}{2} a \sin (m+1)\,\theta + \tfrac{1}{2} a \sin (m-1)\,\theta \\ & - \tfrac{1}{2} b \cos (n+1)\,\theta + \tfrac{1}{2} b \cos (n-1)\,\theta \\ & + \tfrac{1}{2} c \sin (p+1)\,\theta + \tfrac{1}{2} c \sin (p-1)\,\theta \\ & - \tfrac{1}{2} d \cos (q+1)\,\theta + \tfrac{1}{2} d \cos (q-1)\,\theta \\ & + \tfrac{1}{2} e \sin (v+1)\,\theta + \tfrac{1}{2} e \sin (v-1)\,\theta \\ & - \tfrac{1}{2} f \cos (w+1)\,\theta + \tfrac{1}{2} f \cos (w-1)\,\theta + r \cos (\theta - \varphi) \end{aligned}$$

$$\begin{aligned} y = {} & \tfrac{1}{2} a \cos (m+1)\,\theta - \tfrac{1}{2} a \cos (m-1)\,\theta \\ & + \tfrac{1}{2} b \sin (n+1)\,\theta + \tfrac{1}{2} b \sin (n-1)\,\theta \\ & + \tfrac{1}{2} c \cos (p+1)\,\theta - \tfrac{1}{2} c \cos (p-1)\,\theta \\ & + \tfrac{1}{2} d \sin (q+1)\,\theta + \tfrac{1}{2} d \sin (q-1)\,\theta \\ & + \tfrac{1}{2} e \cos (v+1)\,\theta - \tfrac{1}{2} e \cos (v-1)\,\theta \\ & + \tfrac{1}{2} f \sin (w+1)\,\theta + \tfrac{1}{2} f \sin (w-1)\,\theta - r \sin (\theta - \varphi) \end{aligned}$$

If the starting phases are not 0°, they may be inserted as before (233). What was said about the pen rotating in five circles when there was but one component to X and Y in the rectangular curve, is now seen to be generalized for any number of components, each of which adds two circles to the motion. The direction is as before alternately inverse and direct.

Special Cases

These special cases will here be confined mostly to those with one component each in X and Y in the original rectangular curve, that is, to Lissajou curves, drawn on a rotating disk.

236. *First.* Let $m = n$. When the starting phases ς and η are both 0°, the rectangular curve is a straight line (223), which may be drawn by the Y component alone. This case will be treated as a Sine - Polar, to be seen later (241).

Let η alone be 0°, and let ξ have any value. The Lissajou curve is then an ellipse (223). The General Equation (233) then becomes

$$\begin{aligned} x &= \tfrac{1}{2} a \sin \left((m+1)\,\theta + \xi \right) + \tfrac{1}{2} a \sin \left((m-1)\,\theta + \xi \right) \\ &\quad - \tfrac{1}{2} b \cos (m+1)\,\theta + \tfrac{1}{2} b \cos (m-1)\,\theta + r \cos (\theta - \varphi) \\ y &= \tfrac{1}{2} a \cos \left((m+1)\,\theta + \xi \right) - \tfrac{1}{2} a \cos \left((m-1)\,\theta + \xi \right) \\ &\quad + \tfrac{1}{2} b \sin (m+1)\,\theta + \tfrac{1}{2} b \sin (m-1)\,\theta - r \sin (\theta - \varphi). \end{aligned}$$

By using the formulae for the sine and cosine of the sum of two angles as models, it is found that

$$\begin{aligned} x &= \tfrac{1}{2} (a \sin \xi - b) \cos (m+1)\theta + \tfrac{1}{2} (a \sin \xi + b) \cos (m-1)\,\theta \\ &\quad + \tfrac{1}{2} a \cos \xi \left[\sin (m+1)\,\theta + \sin (m-1)\,\theta\right] + r \cos (\theta - \varphi) \\ y &= -\tfrac{1}{2} (a \sin \xi - b) \sin (m+1)\,\theta + \tfrac{1}{2} (a \sin \xi + b) \sin (m-1)\,\theta \\ &\quad + \tfrac{1}{2} a \cos \xi \left[\cos (m+1)\,\theta - \cos (m-1)\,\theta\right] - r \sin (\theta - \varphi). \end{aligned}$$

Here the coefficients are again all constant, and the angles respectively the same.

237. *Secondly.* Let $m = n$, $a = b$, $\xi = -90°$, then

$$x = -a \cos (m+1)\,\theta + r \cos (\theta - \varphi)$$
$$y = a \sin (m+1)\,\theta - r \sin (\theta - \varphi)$$

When $\xi = +90°$.

$$x = a \cos (m-1)\,\theta + r \cos (\theta - \varphi)$$
$$y = a \sin (m-1)\,\theta - r \sin (\theta - \varphi)$$

So that $\xi = -90°$ gives $m+1$ and $\xi = +90°$ gives $m-1$.
As φ is now only an initial phase, it may be dropped by making $\beta = 0$ (231).

238. *Thirdly.* Let $m = n$, $a = b$, $r = a\,(m+1)$, $\xi = -90°$.

then $x = -a \cos (m+1)\,\theta + a\,(m+1) \cos \theta$
$y = a \sin (m+1)\,\theta - a\,(m+1) \sin \theta$

which is the general equation for an epicycloid of m cusps.

Fourthly, Let $m = n$, $a = b$, $r = a\,(m-1)$, $\xi = +90°$.

then $x = a \cos (m-1)\,\theta + a\,(m-1) \cos \theta$
$y = a \sin (m-1)\,\theta - a\,(m-1) \sin \theta$

which is the general equation for a hypocycloid of m cusps.

Here these cycloids are treated as rectangular-polar curves, in (226) they were taken as rectangular ones only. See Chapter V for more details.

239. *Fifthly.* Figs. 753A, 923, and 936 have two components in each axis. The last is a variable circle with its center moving in a circle, and consists of two compound cycles of Fig. 934 drawn on a disc. Its equation is found from (235) by making $a = b = c = d = 1$, $e = f = 0$, $m = n = 45$, $p = q = 44$, $r = 2.7$, $\varphi = 0°$. Then

$$x = \tfrac{1}{2}\,(\sin 46\,\theta + \sin 44\,\theta - \cos 46\,\theta + \cos 44\,\theta + \sin 45\,\theta + \sin 43\,\theta - \cos 45\,\theta + \cos 43\,\theta) + 2.7 \cos \tfrac{1}{2}\,\theta$$
$$y = \tfrac{1}{2}\,(\cos 46\,\theta - \cos 44\,\theta + \sin 46\,\theta + \sin 44\,\theta + \cos 45\,\theta - \cos 43\,\theta + \sin 45\,\theta + \sin 43\,\theta) - 2.7 \sin \tfrac{1}{2}\,\theta.$$

IV. Sine-Polar Curves

In this class of curves* the pen moves on or parallel to the Y axis while the disk rotates underneath. This is a much simpler class than the preceding one, and is the one generally used on a disk. The mathematics may be deduced from the General Equation (233) by making the amplitude $a = 0$. This then becomes

241. *First,* when the pen moves parallel to a radial line

$$x = -\tfrac{1}{2}b \cos (n + 1)\,\theta + \tfrac{1}{2}b \cos (n - 1)\,\theta + r \cos (\theta - \varphi)$$
$$y = \tfrac{1}{2}b \sin (n + 1)\,\theta + \tfrac{1}{2}b \sin (n - 1)\,\theta - r \sin (\theta - \varphi).$$

A simpler equation may be obtained by making $a = o$ in (231)

$$x = (b \sin n\,\theta + \beta) \sin \theta + \alpha \cos \theta$$
$$y = (b \sin n\,\theta + \beta) \cos \theta - \alpha \sin \theta.$$

Then if ϱ = radius vector $= \sqrt{x^2 + y^2}$, the polar equation, when the starting phase η is inserted, is

$$\varrho^2 = [b \sin (n\,\theta + \eta) + \beta]^2 + \alpha^2.$$

If the curve is to be cuspidal, see the Appendix on Cuspidal Rosettes.

When there are many components the equation is

$$\varrho^2 = [b \sin (n\,\theta + \eta) + d \sin (q\,\theta + v) + f \sin (w\,\theta + \lambda) + \ldots + \beta]^2 + \alpha^2.$$

See the Appendix for Cuspidal Envelope Rosettes.

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* The term, Sine-Polar, may not be perfectly correct, but no better expression could be found.

242. *Secondly,* When the pen moves on a radial line, and has but one component, both a and α are zero, and the equation is

$$\varrho = \beta + b \sin (n\,\theta + \eta)$$

When $\eta = 0°$ $\quad \varrho = \beta + b \sin n\,\theta$

when $\eta = \pm 90°$ $\quad \varrho = \beta \pm b \cos n\,\theta$

the familiar equations of a rosette.

Special Cases of Rosettes

243. *First,* when $n = 1$, $\beta = b$,

$$\varrho = b\,(1 + \sin(\theta + \eta)\,).$$

This is a *cardioid,* a rosette of one lobe, for all values of η. The simpler cases are

when $\eta = 0°$	$\varrho = b\,(1 + \sin\theta)$
$\eta = 90°$	$\varrho = b\,(1 + \cos\theta)$
$\eta = 180°$	$\varrho = b\,(1 - \sin\theta)$
$\eta = 270°$	$\varrho = b\,(1 - \cos\theta)$.

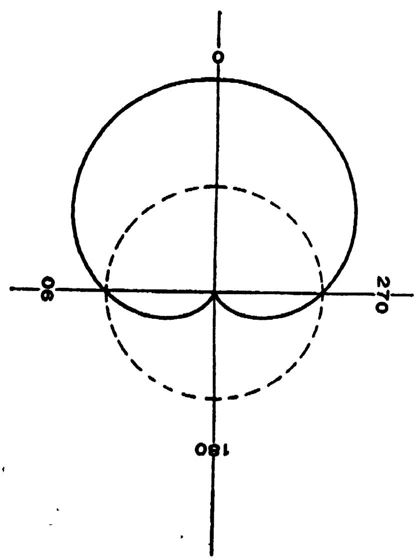

Fig. 243. The Cardioid.

Fig. 243 illustrates the four cases mentioned, as well as all others, and shows how the axis of the cardioid swings round at a uniform rate

in a clockwise direction as η grows. The figure must be held so that the respective value of η is on top or on $+Y$, b being the radius of the circle.

244. *Secondly,* When $n = 1$, $\beta = 0$,

$$\rho = b \sin (\theta + \eta)$$

a circle for all values of η. The principal ones are

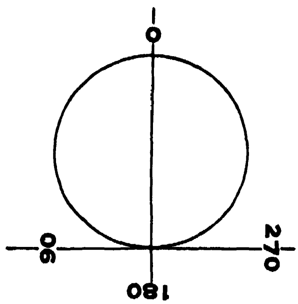

FIG. 244. The Circle.

when $\eta = 0°$	$\rho = b \sin \theta$
$\eta = 90°$	$\rho = b \cos \theta$
$\eta = 180°$	$\rho = - b \sin \theta$
$\eta = 270°$	$\rho = - b \cos \theta$

The axis, or diameter, of the circle thus swings round like the axis of a cardioid for increasing values of η. The circle may thus be classed as a rosette.

245. *Thirdly,* the *Spiral of Archimedes,* $\rho = a\,\theta$, may be classed under rosettes, $\rho = \beta + b \sin n\,\theta$, by making $\beta = 0$ and n very small. If $n = 1/360$ the pen in one revolution will have advanced from zero only the length of $b \sin 1°$. Then since sines of small angles are equal to their arcs, as far as a graphic curve is concerned, by making n sufficiently small, $n\,\theta$ may be written for $\sin n\,\theta$. The microscopic value of n may then be offset by a large one of b, so that $bn = a$ may be any quantity. Therefore $\rho = bn\,\theta = a\,\theta$.

The Spiral of Archimedes is drawn by setting the pen in the center of the disk and winding up the cord on a shaft that turns very slowly while the disk is rotating moderately fast. When the pen is made to approach the center of the disk, it reverses the direction of rotation of the spiral. See Frontispiece 245B.

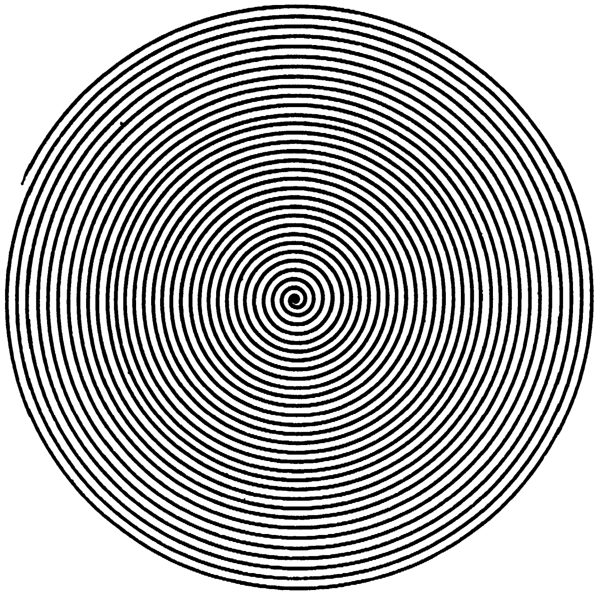

FIG. 245A. The Spiral of Archimedes, direct.

246. *Fourthly.* When there are many components, the equation is

$$\varrho = \beta + b \sin (n\,\theta + \eta) + d \sin (q\,\theta + \nu) + f \sin (w\,\theta + \lambda) + \ldots$$

When two components are used with equal amplitudes and with slightly different periods, a new class of rosettes may be drawn, that of *Envelope Rosettes* (see Appendix). Here the rosette is, as it were a boundary (216 note) to the complex curve. Thus for the *Septifolium* (Appendix Fig. 7) there were two radial components to Y with 16×7 and 15×7 cycles, while the disk rotated 15 times. The initial phases were both 90°, so that the pen was started at the center of the disk. Then, as $b = d = 1$, $\beta = 2$, $\eta = \nu = 90°$, the equation is

$$\varrho = 2 - \cos 7\,\theta - \cos \frac{16}{15}\, 7\,\theta.$$

When the cosine equation is used, that is, when the initial phases are 90°, and $\beta = 2b$, as in the septifolium just mentioned, the rosette is called an *inner* one, because it is on the inside of the complex curve. But when the initial phases are 0°, and β is zero, the sine equation results, and the rosette is an *outer* one or forms the outer boundary. Thus for Fig. 4 in the Appendix

$$\varrho = \sin 2\theta + \sin \frac{16}{15}\, 2\theta.$$

The inner and outer rosettes are always present in every such figure. When the factors 7 and 2 to θ in the above equations are called n, each of the rosettes has $2n$ lobes in principle. When n is odd, the lobes are coincident in pairs for the outer envelopes, and crossed equiangularly when n is even.

247. As sine curves are periodic in character, the mechanism of a machine may sometimes make it possible to bend into a complete circle that length of the X axis which measures a complex period. Care should be taken, however, not to place the pen too near the center of the disk and thus make the radius β of this circle too small, because then the inner parts of the curve will become cuspidal in character. Much less should the pen ever pass beyond this center, as then the very shape of the curve and its successive parts may easily become unrecognizable. Fig. 247 illustrates the idea. The left curve A is one

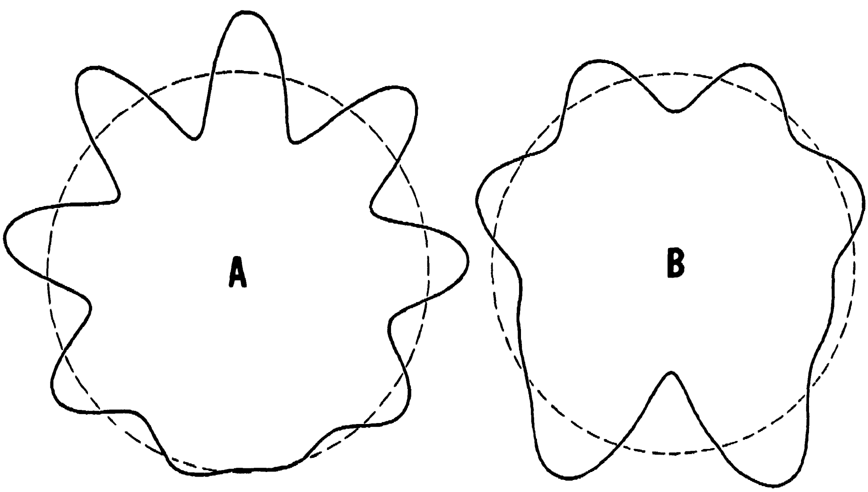

FIG. 247. Two Compound Circular Sine Curves.

complex period of the curve $y = \sin 8\theta + \sin 9\theta$, and the right B one of the curve $y = \cos 4\theta + \cos 5\theta + \cos 6\theta + \cos 8\theta$. Musicians will readily see the allusion. Such curves might be called rosettes with one complex lobe (246). Their polar equations would be $\rho = \beta$ plus or minus the values just given for y, plus when a positive direction is away from the center, and minus when it is towards it.

V. Rectangular-Sine Curves

251. By Rectangular-Sine Curves* is meant that class in which the pen describes a rectangular curve with one or more components in both X and Y, while the paper is drawn under it with uniform speed parallel to the X axis.

252. With but one component in each axis, the equation is

$$x = a \sin (m\theta + \xi) + g\theta$$
$$y = b \sin (n\theta + \eta).$$

Here, as usual, a and b are the amplitudes, m and n the periods, ξ and η the initial phases, and g a constant dependent upon the linear speed of the paper. The equation may be compounded in the ordinary way, that is, by merely adding $g\theta$ to the abscissa of any rectangular curve (221).

Special Cases

253. When $a = b$, $m = n = 1$, and ξ and η differ by 90°, the pen describes a circle, which moves forward with the speed $g\theta$. Thus in Fig. 932

$$x = -\cos\theta + \tfrac{1}{48}\theta$$
$$y = \sin\theta.$$

The constant g is found by counting the number of revolutions, here 48, that the circle makes in advancing the distance 2π, its radius being unity. If it made only one (255), g would be equal to 1. Hence the circle advances with 1/48 the speed, and $g = 1/48$. The equation may also be written

$$x = -\cos 48\,\theta + \theta$$
$$y = \sin 48\,\theta.$$

254. When $\xi - \eta$ has any value except 0° and 90° (223), the curve is a progressive ellipse, as in Fig. 942A.† Its equation is

$$x = \sin (48\,\theta - 50°) + \theta$$
$$y = \sin 48\,\theta.$$

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* The expression, Rectangular-Sine, may not be unobjectionable, but no better one could be found. In popular language these would be called ribbon curves.

† In the frontispiece.

Reversing the sign of ξ makes the ellipse lean the other way, 942B.* The rotation of the pen is now direct, while it is inverse in the preceding figure (223).

When $\xi = \eta$ (223) the figure becomes a slanting sine curve, as in Fig. 922B, just as if oblique axes were used, the equation being

$$x = \sin 48\,\theta + \theta$$
$$y = \sin 48\,\theta.$$

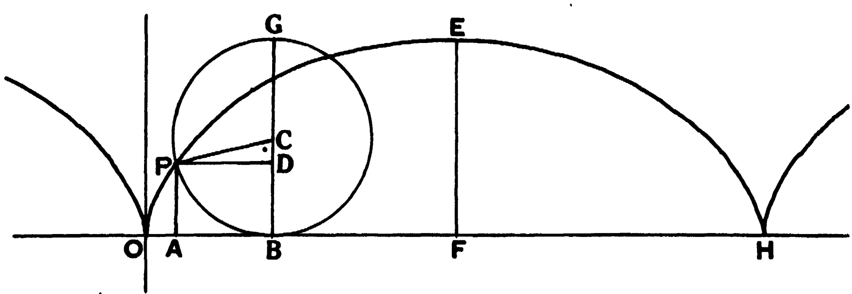

FIG. 255. The Common Cycloid.

255. Let $a = b$, $m = n = 1$, $\xi = 180°$, $\eta = -90°$, $g = a$, and let the origin be shifted from the center of the circle to its lowest point, o, $-a$, then

$$x = a\,\theta + a \sin(\theta + 180°) = a\,(\theta - \sin\theta)$$
$$y = a + a \sin(\theta - 90°) = a\,(1 - \cos\theta)$$

which is the parametric equation of a common cycloid. In order to eliminate θ, it is seen that

$$\cos\theta = \frac{a - y}{a},\quad \sin\theta = \frac{\sqrt{2ay - y^2}}{a},\quad \text{vers}\,\theta = 1 - \cos\theta = y/a$$

$$\theta = \text{vers}^{-1}(y/a)$$

then $$x = a\,\text{vers}^{-1}(y/a) \mp \sqrt{2ay - y^2}$$

which is the ordinary Cartesian form of the equation of a common cycloid (512).

256. When m and n are not equal but have a close ratio, the figure presents a heavy sinusoid, as it were, in lattice work, 959A.* The equation is†

* Frontispiece.

† For the value of g see (216).

$$x = \sin 44\,\theta + 6\,\theta$$
$$y = \sin 45\,\theta.$$

This equation differs from 253 and 254 in that the periods are unequal, and from 216 in that y has given one of its components to x.

257. In the cases just given, there is only one component in each axis. Fig. 959B* has two in X and one in Y. The sinusoid in 959A* is now replaced by a cycloid. The equation is

$$x = \tfrac{5}{8}\,(\sin 45\,\theta + \sin 44\,\theta) + 6\,\theta$$
$$y = \cos 45\,\theta$$

258. Fig. 947 shows two components in each axis. The pen drew two unequal circles in opposite directions with unequal speed. The appearance was as if it drew an ellipse which rotated about its center while this center travelled in a straight line. The equation is

$$x = \cos 45\,\theta - \tfrac{1}{2}\cos 44\,\theta + 6\,\theta$$
$$y = \sin 45\,\theta + \tfrac{1}{2}\sin 44\,\theta.$$

VI. Stereoscopic Curves

261. The only difference between the two plane curves of a stereoscopic pair† is a slight change in the initial phase of one or more of the components. In the rectangular curve

$$\begin{aligned} x &= a \sin (m\,\theta + \xi) \\ y &= b \sin (n\,\theta + \eta) \end{aligned} \qquad \text{or} \qquad \begin{aligned} x &= a \sin m\,\theta \\ y &= b \sin n\,\theta \end{aligned}$$

each of the co-ordinates, x and y, is the projection of a circular motion. With a third axis Z, at right angles to the other two, it is evident that x comes from a rotation in the XZ plane, and y from one in the YZ plane. As the axis of X is usually kept horizontal, parallel to the line joining the eyes, in a stereoscopic pair of figures it is only the rotation about the Y axis that can be seen, so that the angle $m\,\theta$ must be changed, or x must change. The eyes then see different pictures. A change in y or in $n\,\theta$ is practically a change in x only, because the y of a point in one figure will sooner or later have the same value it has in the other. Hence, while in a rectangular curve it is x that is to be dephased in principle, in practice it makes no difference whether it is x or y.

* Frontispiece.

† If the reader is not familiar with stereoscopic harmonic curves, it will be necessaty for him to first read Chapter VII.

262. This change of phase, or this parallactic turn, of x is not to be written in a second equation, because there is a better way of taking care of it. As x equals $a \sin m\, \theta$, z must be equal to $a \cos m\, \theta$, on account of the circular motion. Therefore the equation of a two-dimensional plane curve

$$x = a \sin m\, \theta$$
$$y = b \sin n\, \theta$$

becomes, when the same curve is seen stereoscopically or in three dimensional shape

$$x = a \sin m\, \theta$$
$$y = b \sin n\, \theta$$
$$z = a \cos m\, \theta$$

which is evidently cylindrical.

263. When the plane curve is turned at right angles, x and y exchange places. But the horizontal co-ordinate is the only one whose change of phase can be observed. This is true also for any number of components, and explains why turning the figures at right angles produces a different solid. (Figs. 713, 717, 742.)

264. In polar curves, similarly, the sines and cosines of the x terms (233) are replaced respectively by cosines and sines in z. The last term in x, $r \cos (\theta - \varphi)$, however, does not enter z, because this polar rotation is in the XY plane which cannot affect z.

With one component in each axis, the figure resembles a torus. It is a true torus only when $a = b$, and $m = n$. When there are two or more components in each axis, the figure is somewhat like a torus, but then lines run through it from side to side and do not remain on the surface.

265. In polar curves, when the pen does not reach the center, the stereoscopic figures, as said, somewhat resemble a torus and there is apt to be much sameness in this class. There is even greater monotony in rectangular curves when X has only one component. The finest and most varied figures are the polar ones in which the pen passes the center, and the rectangular ones in which X and Y have each two or more components.

VII. Curves With Infinite Branches

271. In harmonic motion the excursions of the pen are evidently confined within the sum of the amplitudes of the components. The curves

traced cannot therefore in principle have infinite branches except perhaps in the case of Sine Curves or Rectangular-Sine Curves, which may proceed without limit parallel to the X axis. Only two or three exceptions have been found, and these are more accidental or apparent than real. One is the parabola. The harmonic equation is

$$x = \sin(\theta + \xi),\ y = \sin(2\theta + \eta)$$

in which either $\xi = 0°$ and $\eta = \pm 90°$, or both ξ and η are $\pm 90°$. When θ is eliminated, the Cartesian equation is $2x^2 = 1 \mp y$ (225). While only corresponding values of x and y less than unity are actually traced by the pen, larger values up to infinity also satisfy the equation. The same may be said of the straight line $y = bx/a$ (223). The Spiral of Archimedes (245), $\rho = a\,\theta$, may be noted in the third instance, because θ may grow indefinitely.

272. The hyperbola is not a harmonic curve, as is evident from its parametric equation,

$$x = a \sec\theta \qquad\qquad y = b \tan\theta$$

which shows that one of the co-ordinates is always greater than its corresponding amplitude, and the other also when $\theta > 45°$.

VIII. Curves That Are Not Harmonic

281. While curves with infinite branches cannot be harmonic, it by no means follows that all finite closed curves must be so. A tempting case is the lemniscate, whose equation, $\rho^2 = \cos 2\,\theta$, will allure every mathematician like an ignis fatuus, the more so, because the curve looks like 743B, except that the amplitude of the Y component is less than that of the X.

282. The test, as to whether a given curve is harmonic, consists in reducing its equation to the parametric form (211), and seeing whether this can be expressed in sines and cosines.

Summary of Chapter II

The Notation:

a, b, c, d, e, f, . . . (in the first half of the alphabet) are amplitudes.

m, n, p, q, v, w, . . . (generally in the second half of the alphabet) are coefficients of the unit angle θ, and denote periods or speed.

θ is the unit or variable angle, in terms of which all others are generally expressed. When a disk is used, θ is usually the angle turned through by the disk.

$\xi, \eta, \mu, \nu, \varkappa, \lambda, \ldots$ (Greek letters) denote starting phases.

α, β, the co-ordinates of the center or origin of the rectangular curve as measured from the center of the disk.

$r = \sqrt{\alpha^2 + \beta^2}$ = distance between these two centers.

φ, the position angle, so that $\alpha = r \cos \varphi$, and $\beta = r \sin \varphi$.

x, y, z, co-ordinates of a point.

ϱ, the radius vector, or distance of a point from the center of the disk.

There Are Six Classes of Curves

(211) I. SINE CURVES, in which there are one or more components in the Y axis only, while the paper is drawn along the X axis with uniform rectilinear speed. Meaning of parametric, Cartesian, and polar equations.

(212) The period, (213) the amplitude, (214) the initial phase.

(216) General Equation.

$$x = \theta$$
$$y = b \sin (n\theta + \eta) + d \sin (q\theta + \nu) + f \sin (w\theta + \lambda) + \ldots$$

Application.

(211) II. RECTANGULAR CURVES, in which there are one or more components in both the X and Y axes, directly attached to the pen, with the paper stationary, or there are components in Y only for the pen, while those in X are given to the paper in the reverse direction.

(222) General Equation.

$$x = a \sin (m\theta + \xi) + c \sin (p\theta + \mu) + e \sin (v\theta + \varkappa) + \ldots$$
$$y = b \sin (n\theta + \eta) + d \sin (q\theta + \nu) + f \sin (w\theta + \lambda) + \ldots$$

SPECIAL CASES. (223) *First.* $c, d, e, f \ldots$ zero.

$$x = a \sin (m\theta + \xi) \qquad \textit{Lissajou}$$
$$y = b \sin (n\theta + \eta) \qquad \textit{Curves.}$$

When $m = n$, and $\xi = \eta = 0°$, $y = bx/a$, *Straight Line.*

When $m = n$, $\eta = 0°$, $x = a \sin (\theta + \xi)$, *Ellipse.*

$$y = b \sin \theta.$$

When $m = n$, $a = b$, $\xi - \eta = \pm 90°$.

$$\left.\begin{array}{l} x = \pm a \cos \theta \\ y = a \sin \theta \end{array}\right. \quad \text{or } x^2 + y^2 = a^2, \text{ and } \varrho = \pm a. \quad \textit{Circle.}$$

The rotation of the pen is direct, or anticlockwise, in all Lissajou curves, when X leads in phase, and the reverse when it lags.

(224) *Secondly.* When $m=1,\ n=2,\ a=b,$

$$x=\sin(\theta+\xi) \qquad\qquad y=\sin(2\theta+\eta).$$

When $\eta=0^\circ$ and $\xi=0^\circ$ or $\pm 90^\circ$,

$$y^2=4x^2(1-x^2). \quad \text{Fig. 743B.}$$

(225) When $m=1,\ n=2,\ a=b,\ \xi=0^\circ$ or $\pm 90^\circ,\ \eta=\pm 90^\circ,$

$$2x^2=1\mp y \qquad \textit{Parabola.}$$

(226) *Thirdly.* When the pen draws two circles, whose periods and radii show reciprocity, so that

$$x=\mp m\cos n\theta+n\cos m\theta$$
$$y=m\sin n\theta-n\sin m\theta$$

the curve is a circular cycloid (238).

(227) Other rectangular curves with one or two components in each axis.

III. RECTANGULAR-POLAR CURVES, in which there are one or more components to the pen in each axis, while the disk rotates underneath.

(231). When there is only one component in each axis,

$$x=(a\sin m\theta+\alpha)\cos\theta+(b\sin n\theta+\beta)\sin\theta$$
$$y=-(a\sin m\theta+\alpha)\sin\theta+(b\sin n\theta+\beta)\cos\theta$$

Here the coefficients of $\sin\theta$ and $\cos\theta$ are variable.

(235) When there are many components in each axis, and the coefficients are made constant.

General Equation.

$$\begin{aligned}
x=\ &\tfrac{1}{2}a\sin(m+1)\theta+\tfrac{1}{2}a\sin(m-1)\theta\\
&-\tfrac{1}{2}b\cos(n+1)\theta+\tfrac{1}{2}b\cos(n-1)\theta\\
&+\tfrac{1}{2}c\sin(p+1)\theta+\tfrac{1}{2}c\sin(p-1)\theta\\
&-\tfrac{1}{2}d\cos(q+1)\theta+\tfrac{1}{2}d\cos(q-1)\theta\\
&+\tfrac{1}{2}e\sin(v+1)\theta+\tfrac{1}{2}e\sin(v-1)\theta\\
&-\tfrac{1}{2}f\cos(w+1)\theta+\tfrac{1}{2}f\cos(w-1)\theta+r\cos(\theta-\varphi)\\
y=\ &\tfrac{1}{2}a\cos(m+1)\theta-\tfrac{1}{2}a\cos(m-1)\theta\\
&+\tfrac{1}{2}b\sin(n+1)\theta+\tfrac{1}{2}b\sin(n-1)\theta\\
&+\tfrac{1}{2}c\cos(p+1)\theta-\tfrac{1}{2}c\cos(p-1)\theta\\
&+\tfrac{1}{2}d\sin(q+1)\theta+\tfrac{1}{2}d\sin(q-1)\theta\\
&+\tfrac{1}{2}e\cos(v+1)\theta-\tfrac{1}{2}e\cos(v-1)\theta\\
&+\tfrac{1}{2}f\sin(w+1)\theta+\tfrac{1}{2}f\sin(w-1)\theta-r\sin(\theta-\varphi).
\end{aligned}$$

The starting phases have here been omitted, in order not to render the equations too cumbersome. They may be inserted at any time, so that

$$x = \tfrac{1}{2}\, a \sin\Big((m+1)\,\theta + \xi\Big) + \ldots - \tfrac{1}{2}\, b \cos\Big((n+1)\,\theta + \eta\Big) + \ldots$$

(233) As the coefficients are all constant and multiply respectively the sines and cosines of the same angles, the pen may be said to rotate continually in a series of circles, each component having two of them, and the last one with radius r being about the center of the disk.

(234) The direction of rotation is alternately inverse (or clockwise) and direct (or anticlockwise), the last one being inverse. The zero points of θ vary accordingly.

SPECIAL CASES.

(236) Only one component in each axis. Let $m=n$, $\eta=0°$, $a=b=1$,
$\xi=-90°$, $r=m+1$.

(238) $x=-\cos(m+1)\,\theta+(m+1)\cos\theta$
$y=\sin(m+1)\,\theta-(m+1)\sin\theta$
which is an epicycloid of m cusps.

When $\xi=+90°$, $r=m-1$
$x=\cos(m-1)\,\theta+(m-1)\cos\theta$
$y=\sin(m-1)\,\theta-(m-1)\sin\theta$
which is a hypocycloid of m cusps (226).

(239) Two components in each axis.

IV. SINE-POLAR CURVES in which the pen moves parallel to the Y axis (or to a radius) over a rotating disk.

(241)

$$\varrho^2=\Big(b\sin(n\theta+\eta)+d\sin(q\theta+v)+f\sin(w\theta+\lambda)+\ .\ .\ .\ +\beta\Big)^2+\alpha^2.$$

(242) When the pen moves on a radial line,
$\alpha=0$, $\varrho=\beta+\sin(n\,\theta+\eta)+\ldots$
When $\eta=0°$, $\varrho=\beta+b\sin n\,\theta$
$\eta=\pm 90°$, $\varrho=\beta\pm b\cos n\,\theta$. *Rosettes.*

(245) The *Spiral of Archimedes.* $\varrho=a\,\theta$.

(246) *Envelope Rosettes* are drawn with two components of equal amplitudes but with slightly different periods.

(247) *Circular Sine Curves.*

V. RECTANGULAR-SINE CURVES, in which the pen draws a rectangular curve while the paper moves with uniform speed along the X axis. The equation (252) is the same as for rectangular curves (222), with the addition of $g\,\theta$ to x.

SPECIAL CASES. (253) With only one component in each axis, $a=b$, $m=n=1$, $\xi-\eta=\pm 90°$, the pen draws a progressive circle.

(254) The curve is a progressive ellipse when $\xi-\eta$ is anything but 0° or $\pm 90°$, and it is a progressive slanting line, when $\xi=\eta=0°$.

(255) In the common cycloid, $g=a$, $\xi=180°$, $\eta=-90°$.

(256) When m and n have a close ratio, and there are one or more components in each axis, various surprising figures are produced.

(258) Two components in each axis, e. g. a rotating and advancing ellipse.

VI. STEREOSCOPIC CURVES (261-265) are a pair of curves which differ only slightly in the initial phase of one or more components in X, and when viewed simultaneously, one by the right eye only, and the other by the left eye only, are seen in three dimensions. The equation of the three-dimensional curve is the same as that of one of its plane generatrices, with the addition of the co-ordinate z, which is identical with x except that its sines and cosines are respectively replaced by cosines and sines of the same angles.

(271) VII. CURVES WITH INFINITE BRANCHES. These are impossible in principle, because the excursions of the pen are essentially confined within the sum of the amplitudes of the components. The parabola, straight line, and Spiral of Archimedes are apparent exceptions, because while their Cartesian or polar equations admit of infinite values of x or y or θ, their parametric equations do not, and the pen can trace only finite lengths of the curves.

(281) VIII. CURVES THAT ARE NOT HARMONIC. Not every finite closed curve is harmonic. The test is that its parametric equation can be expressed in sines and cosines.

CHAPTER III

PLOTTING

By plotting is meant locating the principal points of a curve so that the entire curve may then be drawn through them by estimation. It is obvious that drawing curves in this way cannot attain the precision or the beauty that a machine gives, and that it is even a perfectly hopeless task for intricate curves. It can be done however with great success in the case of simple curves (713, 717, etc.), and students often attain great proficiency in this kind of work. When there is no machine at hand, plotting is, of course, the only way to draw a curve. And even in a complex curve, while there is no intention to actually draw it, the method of doing so is worth knowing. Points may then be located from their co-ordinates or phases, and conversely, the position of a point may make its phase or co-ordinates known. While the actual plotting is evidently a graphic procedure, the co-ordinates of points may be found in two ways, either numerically or graphically.

1. The Numerical Method

311. The numerical method of computing the co-ordinates of a point is, of course, the more accurate one. One example of a rectangular-polar curve will be sufficient to illustrate the process.

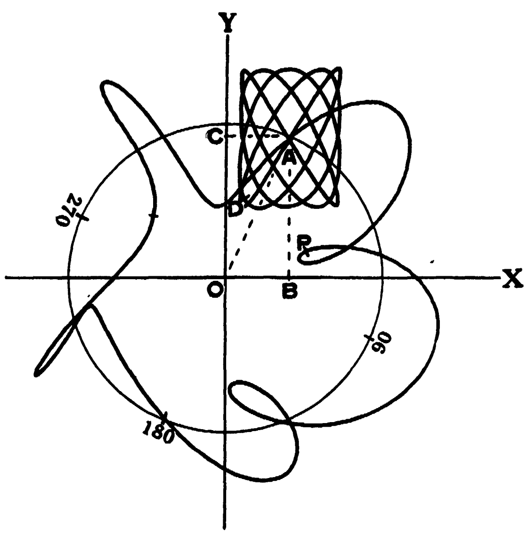

Fig. 311. The Genesis of a Rectangular-Polar Curve

Let the rectangular or Lissajou curve shown in the first quadrant of Fig. 311 be imagined for the moment to be the only thing in the drawing. In its equation

$$x = a \sin m\,\theta$$
$$y = b \sin n\,\theta$$

the amplitudes $a = 1.00$ and $b = 1.44$, the periods $m = 4$ and $n = 5$, and the initial phases ξ and η are 0°.

Then
$$x = 1.00 \sin 4\,\theta$$
$$y = 1.44 \sin 5\,\theta$$

As θ is the independent variable, let $\theta = 60°$ for a definite point.

Then
$$x = 1.00 \sin 240° = -1.00 \sin 60° = -0.87$$
$$y = 1.44 \sin 300° = -1.44 \sin 60° = -1.24.$$

As A is the origin, the point in question is D with its abscissa $x = -0.87$ and its ordinate $y = -1.24$.

This rectangular curve is now to be rotated about O, whose coordinates as measured from A are

$$AC = \alpha = -1.28 \qquad AO = r = 3.22$$
$$AB = \beta = -2.95 \qquad AOX = \varphi = 66°.5.$$

The rotation about O is to be such that while the pen, starting from A, is drawing the rectangular curve once completely, the disk makes one complete direct or anticlockwise rotation, so that relatively, if the disk remained at rest, the rectangular curve, while being drawn, would rotate once about O in a clockwise direction.

When $\theta = 60°$ the disk turns through 60°, and carries the point D to P. Graphically P is quickly located by making the angle $POD = 60°$ and the distance $PO = DO$. Numerically this must be done by formula 233, in which the values of all the constants and of $\theta = 60°$ must be substituted. This then becomes

$$\begin{aligned} x = {} & 0.50 \sin 5 \times 60° + 0.50 \sin 3 \times 60° \\ & - 0.72 \cos 6 \times 60° + 0.72 \cos 4 \times 60° + 3.22 \cos (60° - 66°.5) \\ y = {} & 0.50 \cos 5 \times 60° - 0.50 \cos 3 \times 60° \\ & + 0.72 \cos 6 \times 60° + 0.72 \sin 4 \times 60° - 3.22 \sin (60° - 66°.5) \end{aligned}$$

or

$$\begin{aligned} x = {} & -0.50 \sin 60° + 0.50 \sin 180° \\ & - 0.72 \cos 0° + 0.72 \cos 60° + 3.22 \cos 6°.5 \\ y = {} & 0.50 \cos 60° - 0.50 \cos 0° \\ & + 0.72 \sin 0° - 0.72 \sin 60° + 3.22 \sin 6°.5 \end{aligned}$$

or

$$x = -0.43 + 0.00 - 0.72 - 0.36 + 3.21 = +1.70$$
$$y = +0.25 + 0.50 + 0.00 - 0.62 + 0.36 = +0.49$$

The co-ordinates of P are therefore, when $\theta = 60°$, $x = +1.70$, $y = +0.49$.

312. Fig. 312 illustrates all the component parts of x and y, together with the rotation of the pen in its five circles (233).

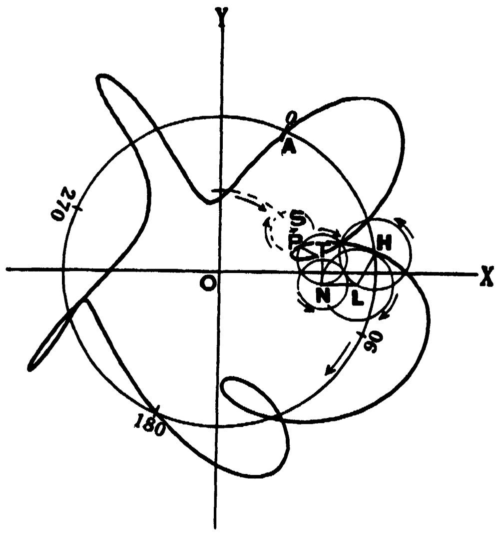

FIG. 312. The Circles in which the Pen moves when drawing a Rectangular-Polar Curve.

The fifth terms in x and y, $+3.21$ and $+0.36$, are the co-ordinates of H, which is on the large circle with radius $r = 3.22$ in the position angle $\varphi - \theta = AOX - AOH = 66°.5 - 60° = 6°.5 = HOX$. This circle rotates (234) in an inverse or clockwise direction.

The fourth terms in x and y, -0.36 and -0.62, are the co-ordinates of L with respect to H. L is on the circle with center at H and radius $\frac{1}{2} b = 0.72$ in the position angle $4 \times 60° = 240°$, and turns direct or anticlockwise.

The third terms, -0.72 and 0.00, are the co-ordinates of N in regard to L. N is on the circle with radius $\frac{1}{2} b = 0.72$, with center at L, position angle $6 \times 60° \doteq 360° = 0°$, and turns inversely. That this position angle of $0°$ should put N to the left of L, instead of to the right, is owing to the fact that the zero of this circle is here where a general one would read $180°$ (234).

The second terms, $+0.00$ and $+0.50$, are the co-ordinates of T with respect to N, in the circle with radius $\frac{1}{2}a = 0.50$, center at N, turning direct, and in the position angle $3 \times 60° = 180°$ as counted from its zero which is where 270° is generally placed.

Finally, the first terms of x and y, -0.43 and $+0.25$, are the co-ordinates of P with regard to T, in the circle $\frac{1}{2}a = 0.50$, center at T, turning inversely, in position angle $5 \times 60° = 300°$ as counted from its zero which is where 90° is generally placed.

Fig. 312 thus illustrates all the fractional constituents of the co-ordinates of the point P where $\theta = 60°$, as well as the five circle of its harmonic motion. The speed of rotation of these circles, beginning with the first which carries the pen, is

	$m+1$,	$m-1$,	$n+1$,	$n-1$,	1,
or in this case	5,	3,	6,	4,	1,

times the period of the disk.

313. *A Second Method.* The point P may also be plotted according to formula 231 with its variable radii. Taking the same case,

$$x = (a \sin m\theta + \alpha) \cos\theta + (b \sin n\theta + \beta) \sin\theta$$
$$y = -(a \sin m\theta + \alpha) \sin\theta + (b \sin n\theta + \beta) \cos\theta$$

becomes

$$x = 0.41 \cos 60° + 1.71 \sin 60° = +0.20 + 1.48 = +1.68$$
$$y = -0.41 \sin 60° + 1.71 \cos 60° = -0.36 + 0.85 = +0.49.$$

The dotted circles in Fig. 312 illustrate these values, the radius of the small circle PS being 0.41 and of the large one OS 1.71. The direction of rotation in both is here inverse or clockwise (223). But this and the radii are, of course, variable, and have the above values and directions only when $\theta = 60°$.

314. Formula 231 may be re-written in this way

$$x = \beta \sin\theta + \alpha \cos\theta + b \sin n\theta \sin\theta + a \sin m\theta \cos\theta$$
$$y = \beta \cos\theta - \alpha \sin\theta + b \sin n\theta \cos\theta - a \sin m\theta \sin\theta.$$

In Fig. 314 the rectangular curve in Fig. 311 has been swung round into the position shown by the dotted rectangle, whose width in the direction of HG is $2a$, and height in the direction HN is $2b$, its middle point A now falling on H, and D on P. $GH = \alpha$, $OG = \beta$, the dotted angles are equal to $\theta = 60°$, $a \sin m\theta = -NP$, $b \sin n\theta = -HN$. Then the re-written formula 231 gives

$$x = OM + RH - UN - NV$$
$$= +2.56 + 0.64 - 1.08 - 0.44 = +1.68$$
$$y = MG - GR - HU + VP$$
$$= +1.48 - 1.15 - 0.62 + 0.76 = +0.47.$$

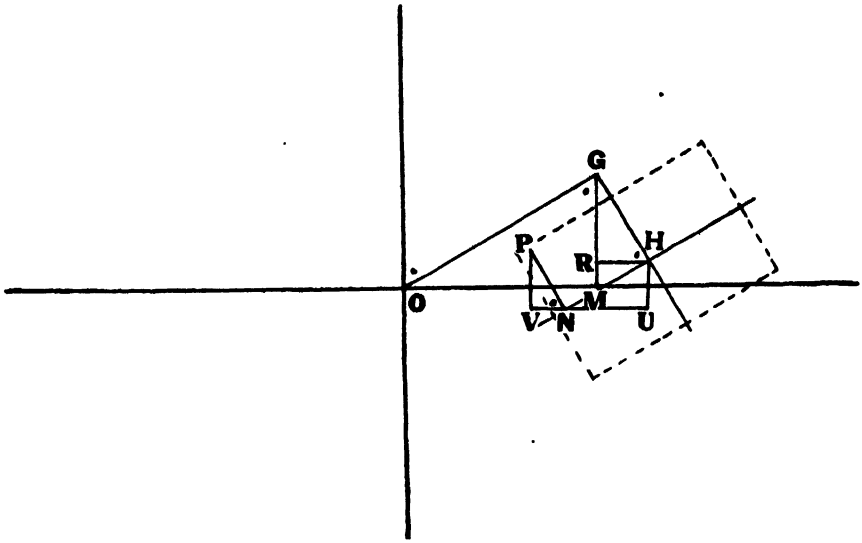

FIG. 314. The Co-ordinates of a Point in a Rectangular-Polar Curve.

315. The following four figures, 315, A, B, C, D, show what becomes of the same rectangular-polar curve that has been illustrated in Fig. 311, when $\alpha = OB$ and $\beta = AB$ are changed, that is, when the pen in phase $\theta = 0°$ is placed on different parts of the disk.

In Fig. A, $\alpha = \beta = 0$, and only the last two terms remain in the preceding equation (314). When $\theta = 60°$,

$$x = -1.08 - 0.44 = -1.52$$
$$y = -0.62 + 0.76 = +0.14.$$

In B $\alpha = a$, $\beta = b$, and

$$x = a\cos\theta + b\sin\theta + a\sin m\theta\cos\theta + b\sin n\theta\sin\theta$$
$$y = -a\sin\theta + b\cos\theta - a\sin m\theta\sin\theta + b\sin n\theta\cos\theta$$

or

$$x = +0.50 + 1.25 - 0.44 - 1.08 = +0.23$$
$$y = -0.87 + 0.72 + 0.76 - 0.62 = -0.01.$$

In C $\alpha = 0$, $\beta = b$,

$$x = b\sin\theta + a\sin m\theta\cos\theta + b\sin n\theta\sin\theta$$
$$y = b\cos\theta - a\sin m\theta\sin\theta + b\sin n\theta\cos\theta$$

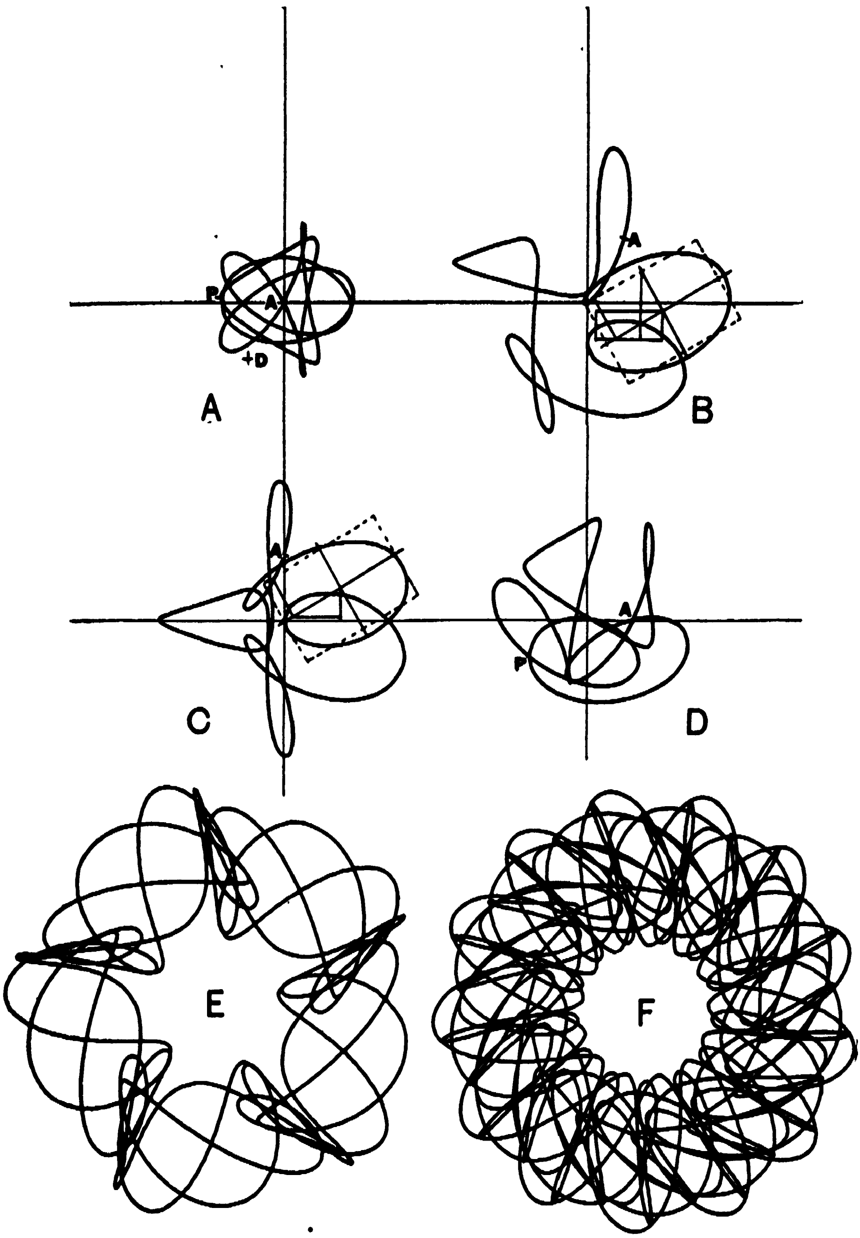

FIG. 315. Changes in a Rectangular-Polar Curve, *A, B, C, D,* as the initial Position of the Pen, and *E, F,* as the number of Cycles is changed.

and when $\theta = 60°$

$$x = +1.25 - 0.44 - 1.08 = -0.27$$
$$y = +0.72 + 0.76 - 0.62 = +0.86.$$

In D, $\alpha = a$, $\beta = 0$, and $x = -1.02$, $y = -0.73$, when $\theta = 60°$.

In Fig. E there are 5 cycles of the rectangular curve in 4 revolutions of the disk, so that in the equations given before m is now equal to $\frac{5}{4}$ 4 or 5 and $n = \frac{5}{4}$ 5.

In Fig. F there are 13 cycles of the rectangular curve in 10 revolutions of the disk, so that $m = \frac{13}{10}$ 4 and $n = \frac{13}{10}$ 5.

II. The Graphic Method

SINE CURVES.

321. The graphic construction of a sine curve hardly needs any explanation. The axis of X is divided off into equal parts (211), which are then numbered in single degrees, in tens, in thirties, etc., according to the scale of the drawing and the accuracy desirable. For the pure sine curve, the sinusoid, 360° must be equal to 2π in length. Ordinates are then erected at the marked points equal to the sines of the arcs, to $\sin x$ (or $\sin \theta$) in the sinusoid, and to $b \sin x$ in any other sine curve.

322. When there are two sine curves given of different periods and amplitudes, and it is desired to graph their resultant, this is done in the way indicated in Fig. 322. The dotted curve has the equation $y = 2 \sin 3\theta$ and the dashed one $y = 3 \sin 2\theta$. When their initial phases are 0°, the full line curve is their resultant. This may be plotted either by drawing the components first, and then adding up their ordinates at the principal points, or independently of these by the two circles in the lower left corner of Fig. 322. The radii of these are the amplitudes 3 and 2. The circumference of the larger has the number of its degrees divided by 2, while those of the smaller are divided by 3. To find the ordinate of the resultant at any angle, the ordinates or sines of these circles at that angle are added algebraically by a dividers, or on the edge of a card.

323. When the initial phases are 40° instead of 0°, so that the Y axis would pass through B in Fig. 322, the two circles just mentioned are

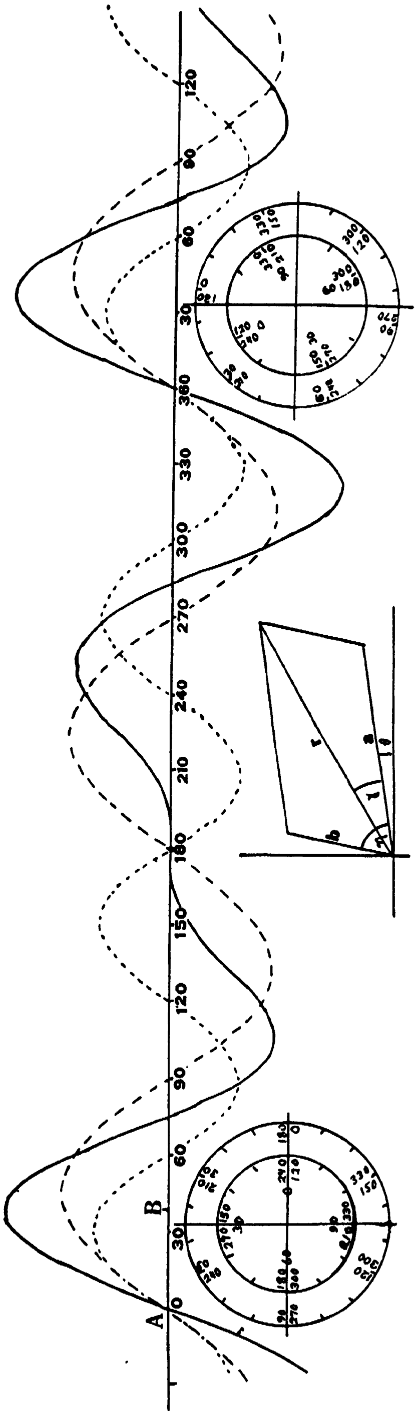

FIG 322. Plotting a Compound Sine Curve.

turned 40° forward on their own graduations, as is shown in the lower right corner. The equations are then $y = 2 \sin 3\ (\theta + 40°)$ and $y = 3 \sin 2\ (\theta + 40°)$. The sum of the ordinates is then found as quickly as it was before. Of course, if one of the curves has another starting phase, its circle must be turned through this. In like manner if the starting phases are minus, the turn is in the other direction.

324. When the periods are equal, the resultant is found still more rapidly. In the small drawing below the middle of Fig. 322, let a represent the amplitude in the component $y = a \sin \theta$, and b the amplitude in $y = b \sin (\theta + \eta)$, in which η is the constant phase difference. When the parallelogram is drawn with a and b as sides and η as their included angle, the resultant of the two will be the diagonal r, and the angle γ between r and a will be its initial phase. This solution may be found much more readily by a plot than by a trigonometric computation. When the angle η between a and b is 90°, $r = \sqrt{a^2 + b^2}$, and the starting phase is such that $\tan \gamma = b/a$. Then as $a = r \cos \gamma$ and $b = r \sin \gamma$, such components of r which are right-angled in phase would be called sine and cosine components (841).

RECTANGULAR CURVES.

331. If it be required to plot graphically the rectangular curve

$$x = a \sin (m\,\theta + \xi)$$
$$y = b \sin n\,\theta$$

when $a = 1.2$, $b = 1.6$, $m = 3$, $n = 2$, $\xi = 20°$, two lines like axes are drawn at right angles, as in Fig. 331, and two circles with the given radii are placed tangent to them in diagonally opposite quadrants. Here the lower circle has the radius $a = 1.2$. Its circumference is divided into 12 parts or 30° intervals, which are numbered from 0 to 24 in two rounds. Its upper point reads 20°, the initial phase. The upper circle with the radius 1.6 is divided into 8 parts or 45° intervals, and is numbered from 0 to 24 in three rounds. Its 0 is on the left end, its initial phase being 0°. A line is now drawn horizontally through 0 on the upper circle, and another vertically through 0 on the lower circle. Their intersection is marked with a dot and the number 0. In like manner such pairs of lines are drawn through equal numbers and correspondingly marked at their intersections. These lines are only construction lines and may be omitted when co-ordinate paper is used. The curve that passes through the numbered dots in consecutive order is then the one desired.

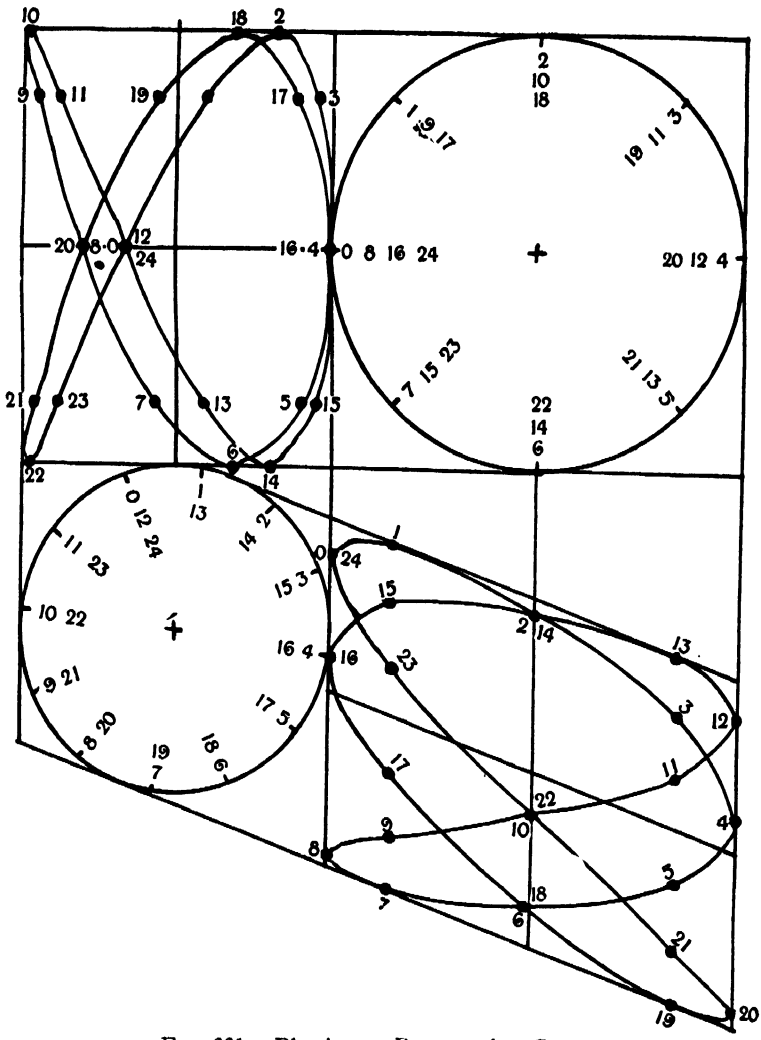

FIG. 331. Plotting a Rectangular Curve.

332. The axes to which the two circles are tangent may make any angle with one another. The curve is drawn however in the same way by lines parallel to the axes. While *a, b, m, n,* are the same as before, for the convenience of the drawing ξ was changed to 130°. Such Lissajou curves are readily drawn in the manner shown, which is, of course, the ordinary one given in books of physics. When there are two or more components in each axis, the resultant of each set should first be found according to Fig. 322.

POLAR CURVES.

341. For plotting polar curves polar co-ordinate paper is very useful. To plot the rosette (242)

$$\varrho = 1.30 + 3.33 \cos 3\,\theta$$
$$\text{or}\quad \varrho = \beta + b \cos n\,\theta$$

a circle is drawn in Fig. 341 with the radius $\beta = 1.30$ and graduated to $5°$ intervals. When $\theta = 0°$, $\cos 3\,\theta = 1$, and $\varrho = \beta + b = 1.30 + 3.33$.

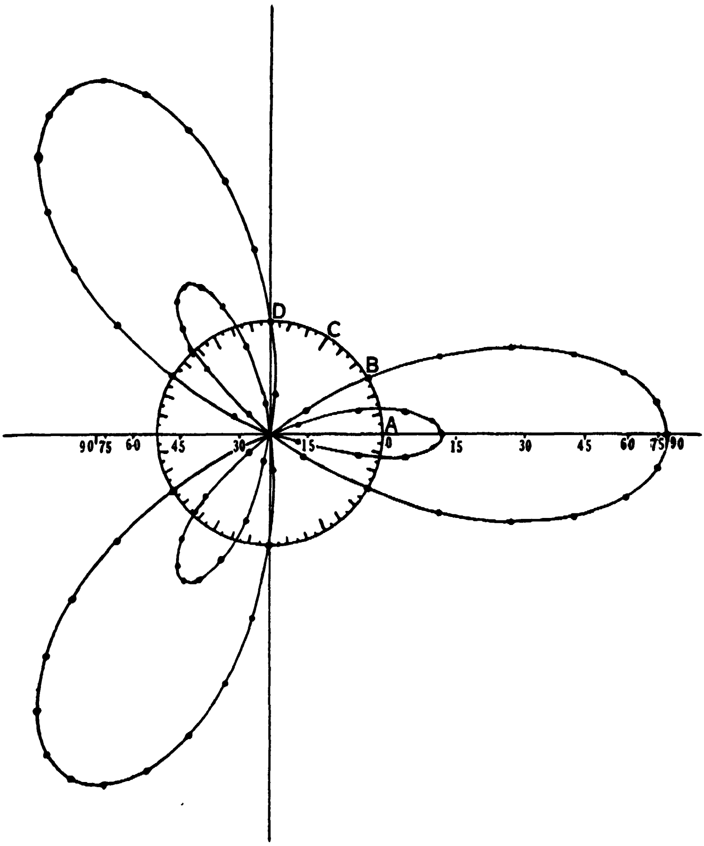

FIG. 341. Plotting a Polar Curve.

When $\theta = 5°$, $\cos 3\,\theta = \cos 15°$, and $3.33 \cos 15°$ is added to $\beta = 1.30$ on the $5°$ line. In order to do this rapidly, the sines of $15°$ and its multiples on the scale of b may be marked on the X axis to the right and left of A.

When $\theta = 30°$, the point of the curve is at B, since $b \cos 3\,\theta$ is then zero. When θ is greater than $30°$, the curve enters the β circle, passes

through its center and emerges on the other side, reaching its maximum distance when $\theta = 60°$. Then it returns, passes through the center again and through D at $90°$, and draws the second large lobe. And so on.

The curve shown has three large and three small lobes, the smaller ones being within the larger, as is always the case when n is odd. The identical rosette may be drawn according to four different equations, the only difference being its position in regard to the axes. Thus

Equations	Axis
$\varrho = \beta + b \cos 3\theta$	OA
$\varrho = \beta - b \sin 3\theta$	OB
$\varrho = \beta - b \cos 3\theta$	OC
$\varrho = \beta + b \sin 3\theta$	OD

When ϱ or b is $\pm$ there are two rosettes, or one double one with six equiangled lobes.

342. Such rosettes take four shapes according to the relative values of β and b. When $b < \beta$, the curve cannot reach the center: it is then called curtate. When $b = \beta$, it touches the center and is cuspidal. When $b > \beta$, the pen passes the center and draws smaller lobes on the other side: the curve is then called prolate. When $\beta = 0$, the lobes are all equal, and the rosette is called equifoliate. The two sets of lobes are coincident when n is odd, so that while there are really $2n$ of them, they appear to be only n. An interesting case of this kind is the circle (244) where $n = 1$.

343. When much plotting of this sort is contemplated, it will be of advantage not only to procure square and polar co-ordinate paper, or to construct models of such, and hold the blank paper over it and over a light, but also to have a variable scale of sines at hand, as shown in Fig. 343, so that a convenient and diversified one may be found rapidly and copied on the edge of a card.

Summary of Chapter III

Plotting a curve is locating its principal points, so that the entire curve may then be drawn through them by estimation.

THE NUMERICAL METHOD (311) locates points by means of their computed co-ordinates. An example is given of a rectangular-polar curve. The circles (312) in which the pen rotates (233) are shown in a diagram. The plotting may also be done (313) by formula 231 with its variable coefficients. The values of α and β are changed

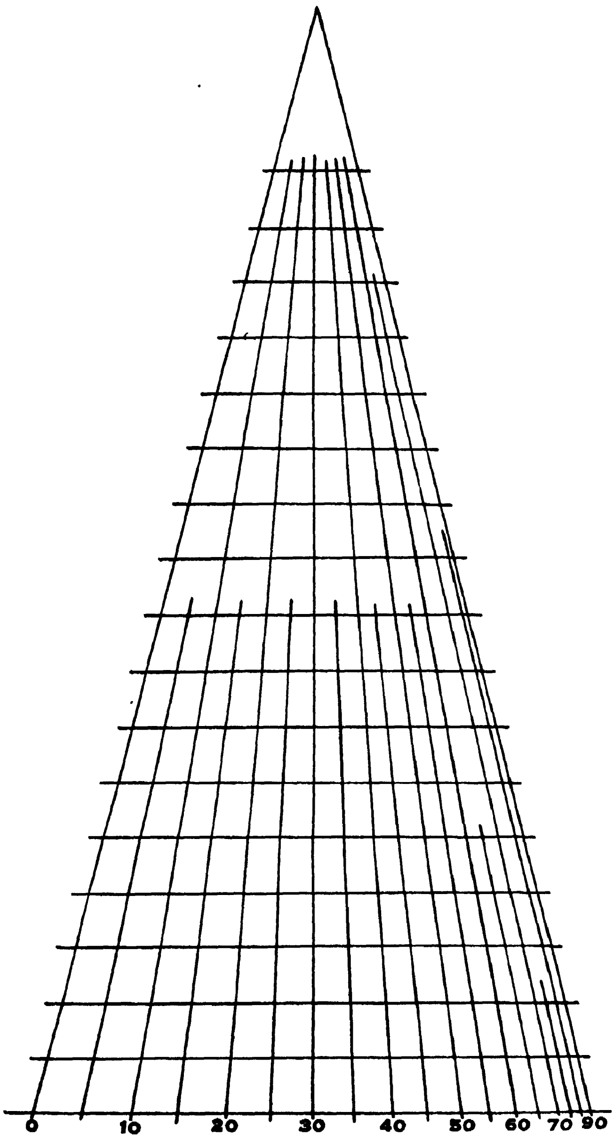

FIG. 343. A Variable Scale of Signs.

(315), that is, the initial position of the pen is changed, and the results shown in several diagrams. The periods are then also altered.

THE GRAPHIC METHOD is illustrated in the plotting of a simple and of a compound sine curve (321-324), of a rectangular curve (331, 332), and of a polar curve (341, 342).

CHAPTER IV

MACHINES

The machines—or instruments, as some may be more properly called—that are used to draw harmonic curves, divide themselves very distinctly into two classes according as they employ pendulums or wheels. They may thus also be styled algebraic and geometric. This division is so pronounced that there does not seem to be any machine in which the two classes are combined, although this might really be done.

The history of these machines, as far as the author was able to learn, has at best only a beginning, and this is confined to the few items given by HAGEN in an article to be mentioned later (413). He says that in 1844 BLACKBURN in Glasgow is reported to have invented a pendulum swinging in two planes at right angles to each other, and that WILLIAM SWAN was the first to use such a pendulum. One of his ways was to let sand run out of a fine nozzle. Another was to let electric sparks flow on to prepared paper and to draw the curves. As these sparks occurred at fixed intervals, their spacing then indicated the velocity of the pendulum.

HUBERT AIRY used a glass tube with a fine nozzle with ink. *Nature*, Aug. 17 and Sept. 7, 1871.

TISLEY perfected AIRY's method in his "Harmonograph," which was made in 1876 in London, and was for sale by TISLEY and SPILLER, Opticians, 172 Brompton Road, London, S. W., at the cost of over 400 marks.

I. PENDULUM MACHINES

411. A pendulum machine consists essentially, as its name implies, of an ordinary pendulum, that is, of a weight suspended from a fixed support and swinging to and fro under the law of gravity. A pen, operated by the swinging weight, then moves with simple harmonic motion (123) and traces the curve. This motion may be compounded by forcing the pendulum to swing in two planes, and by means of a second pendulum, or even of a third and a fourth. No power except that of gravity is ever employed to keep the pendulum in motion, either because this would be too complicated, or rather because it is judged

to be unnecessary on account of the shortness of the time during which the pendulum is used.

Physics teaches that the vibrations of a pendulum are isochronous (124), that is to say, that its period is constant, and that this period is directly as the square root of its length. The weight of the bob has no influence whatever in principle on the period. It is advisable, however, in practice to make the weight heavy, in order that its inertia may overcome the resistance, especially of the air, and thus sustain the vibrations. For the same reason the length should be great, because the slower motion that this generates has the same good effect. The friction, however, while not affecting the period, does shorten the amplitude noticeably during each vibration, so that the pen never retraces its path and the curves are not closed. But as this diminution of the amplitude is uniform, it rather adds to the beauty of the figures and constitutes their principal charm.

412. The simplest kind of a pendulum is a weight hung by a string or wire. This has, however, the disadvantages that the weight may oscillate

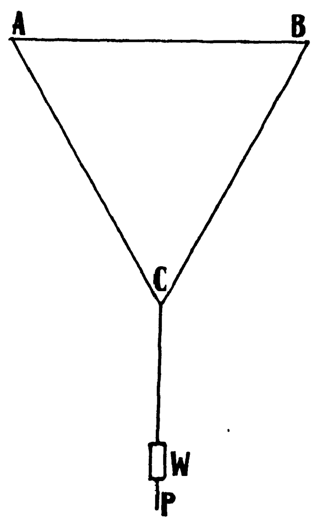

FIG. 412. A very simple Harmonic Pendulum.

on its own axis, the path of the pen may easily become elliptical, and may even in the course of five minutes appear to shift its plane like a Foucault pendulum. All these difficulties may be overcome by bifilar suspensions.

Old textbooks of physics generally mention a pendulum like that outlined in Fig. 412. Here A and B are the points of suspension over

which a looped cord is placed to support the weight W and the pen P. When this contrivance is to be used as a single pendulum, the clasp C is shoved down to W. When it is anywhere above W, the pendulum is a double one, one with the length WC, and the other with the length from W to the line AB. As A, B, C, W, are always necessarily in the same plane, whatever component WC may receive at the start at right angles to AB, will at once be incorporated in the longer pendulum. And finally, the part of the cord below C may also be made bifilar. This intensely simple form of pendulum has been brought to an almost ideal perfection by DOBSON.

THE DOBSON DUPLEX PENDULUM

413. The description of this machine is taken from an article "Ueber die Verwendung des Pendels zur graphischen Darstellung der Stimmgabelcurven" by JOHN G. HAGEN, S. J., now director of the Vatican Observatory, in the *Zeitschrift Für Mathematik und Physik*, 1879, xxiv, 285-303. Published by B. G. TEUBNER, Leipzig.

HAGEN writes that the machine in question was made by JOHN DOBSON, S. J., Professor of Physics and Chemistry in Stonyhurst College, England, where he saw it in the fall of 1877. At the first opportunity he then constructed a similar one for himself.

Fig. 413 outlines the principal details. Two wires proceed from O, where they may be lengthened or shortened, over a fixed pulley A

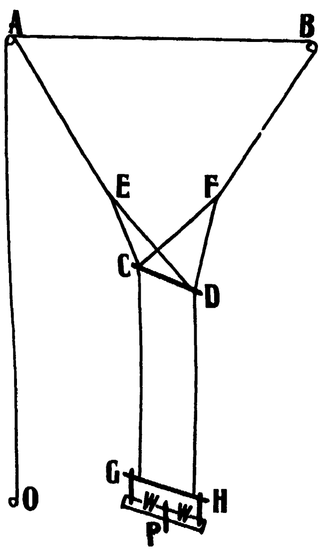

FIG. 413. Dobson's Wire Pendulum.

One then runs down directly to E, and the other over B to F. Here both wires become bifilar and run to the rod CD, from which two other wires descend to the rod GH. At G is a ratchet and crank, so that the wires CG and DH may be wound or unwound on the rod GH. This rod GH is secured by standards to a small table, which carries the pen P and heavy weights $W'W$.

The pendulum is therefore a double one. The first has the length from W to the wire AB, and swings in a north and south plane when the wire AB runs east and west. The second has the length from W to the rod CD, and swings in an east and west plane only. The lengths of the pendulums may be adjusted in advance approximately, their true lengths being found later from the curve after it has been drawn.

As the good performance of the pendulum depends upon its length and weight, HAGEN hung his in the frame of a window, so that it had a length of about 3 meters (or 10 feet), and carried a weight of not less than 12 or 15 kilograms (26 or 33 pounds). The pen consisted of a glass tube with a very short fine nozzle. This last was ground to a correct shape by letting it swing over an oilstone. DOBSON slightly underbalanced his pen on a lever, so that it would touch the paper very gently. When he was about to use it, he placed a rider on the weight end, so as to overbalance and lift the pen off the paper, and then knocked the rider off when the pendulum swung properly.

A good ink to use is any aniline color dissolved in hot water and filtered. The addition of a little gum or glue is of questionable utility. The interior of the pen must be chemically clean, and the ink sucked up into it just before use. The best kind of paper is of the glossy variety. More minute details must be learned by practice.

414. HAGEN gives six illustrations with the ratios 5:6, 4:5, 3:4, 2:3, 5:8, 3:5, which are reproduced in Fig. 414.

THE HOPKINS DUPLEX PENDULUM

421. In his book "Experimental Science," Munn and Co., 1911, New York, Vol. II, pp. 127-129, GEORGE A. HOPKINS describes a simple and effective duplex pendulum machine of his own design. It is very conveniently arranged for ordinary lantern projection, the smoked or colored glass plate being held stationary and vertical. Two pendulums with metal or wooden rods and sliding weights of 12 pounds each, swing from a common pivot O, Fig. 421. The drawing pen is a short needle point P set in a small hole that is drilled into a piece of plain glass twice as large as the plate to be engraved, so that it is almost

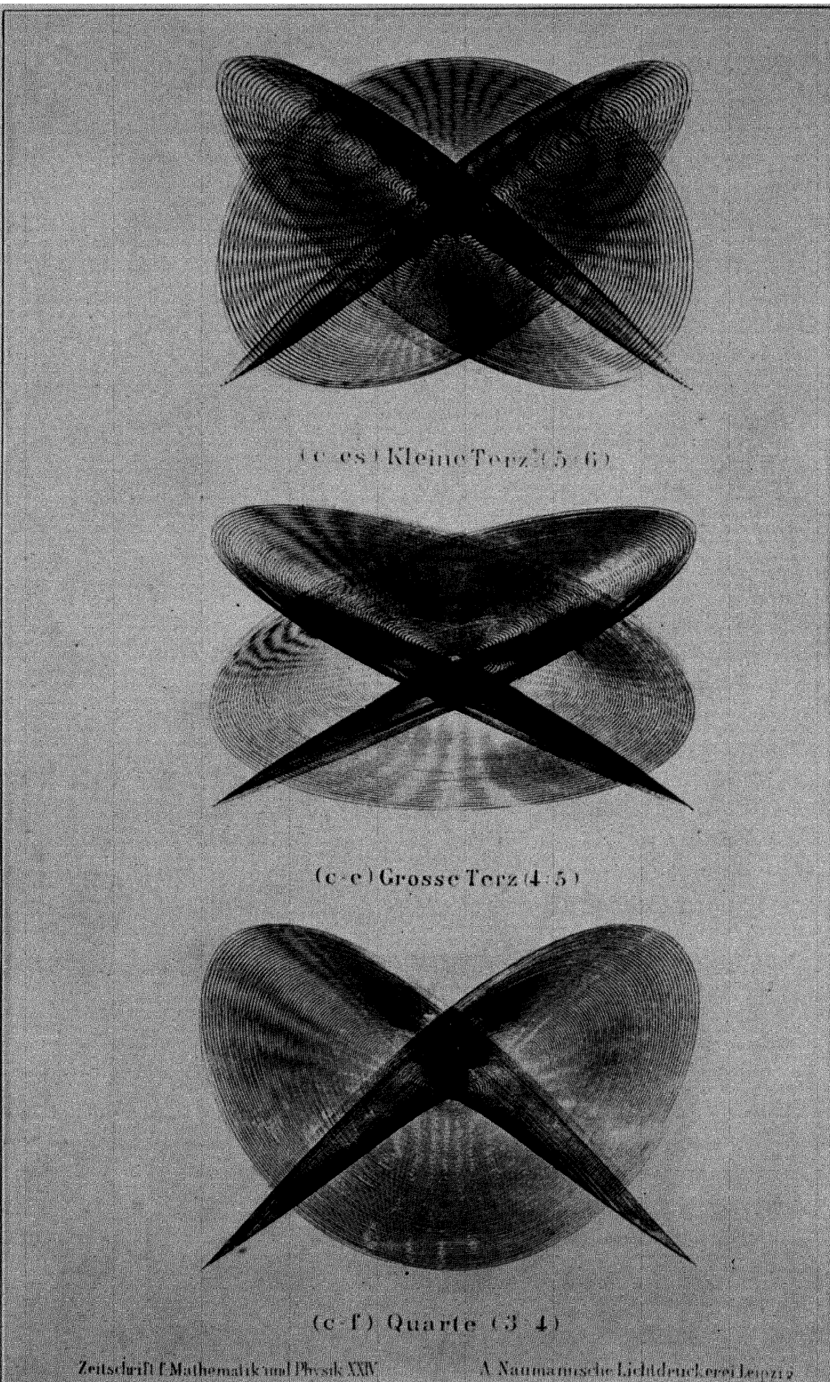

FIG. 414. Figures made by Fr. Hagen with a Dobson Pendulum.

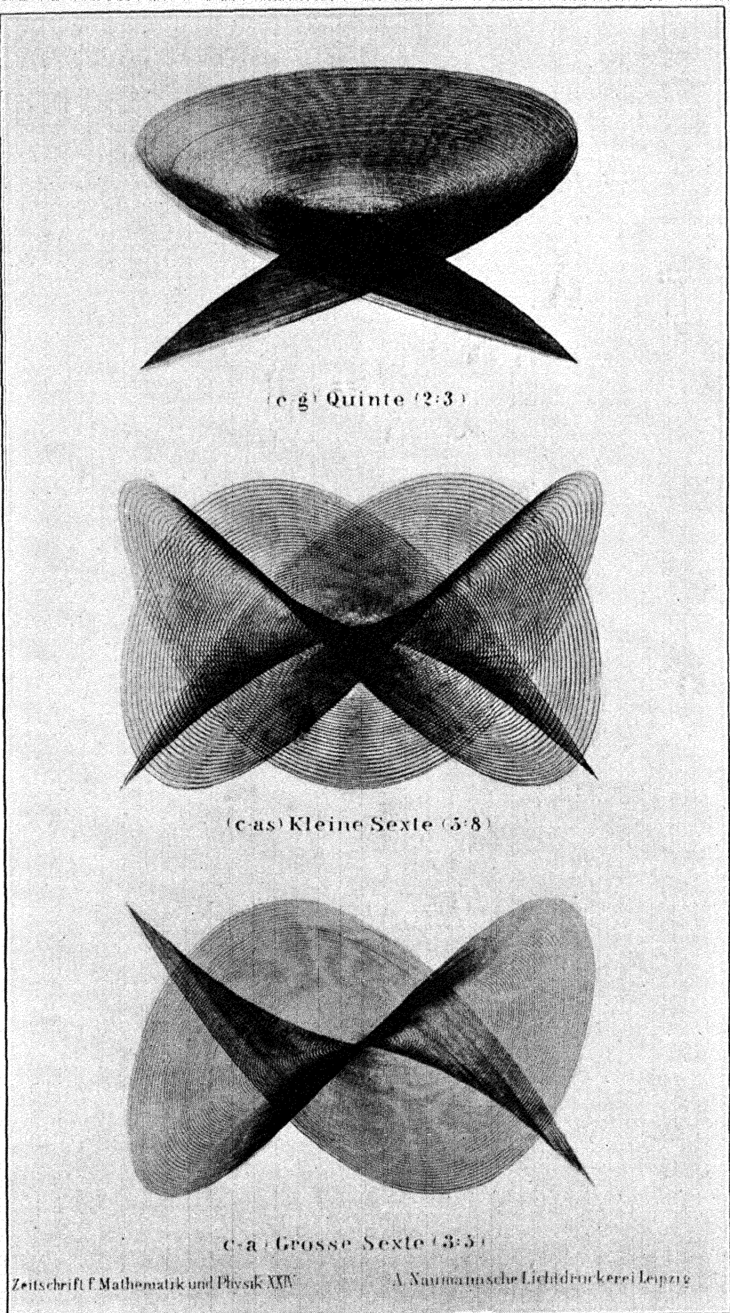

FIG. 414. Figures made by Fr. Hagen with a Dobson Pendulum.

invisible, while the figure it draws seems to grow in some mysterious way. This pen glass *P* is held at the end of a horizontal bar pivoted to one pendulum at *C*, and is extended to carry a balancing weight *D*. While this *A* pendulum gives the tracer a horizontal harmonic motion, the other pendulum *B* gives it a vertical one by means of a short bar

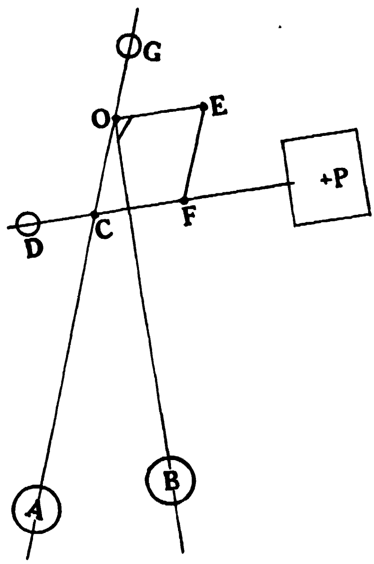

FIG. 421. Hopkins' Duplex Pendulum.

OE fastened at right angles to it and linked to the pen bar *DP* through *EF*. One of the pendulums may have weights of from 2 to 6 pounds *G* attached to it above its pivot *O*. The coloring solution for the plate consists of an aniline color dissolved in collodion, alcohol and water.

THE HOFERER QUADRUPLEX PENDULUM

431. This machine was constructed by MICHAEL J. HOFERER, S. J., and was described by him in the *Scientific American* of April 1, 1899, lxxx, 200, with 20 illustrations. It was afterwards incorporated by GEO. A. HOPKINS in his "Experimental Science," Vol. II, pp. 420-428.

"It consists of a solid table 40 inches in height, four leaden pendulum weights, of 12 pounds each, and capable of being raised or lowered at will," two operating the pen and two the plate; "four ½-inch brass tubes resting upon knife edges and carrying gimbals at the top with steel wires, which are connected hinge-fashion with the needle and the plate holder. This plate holder is suspended from a standard 20 inches in height, and carries a darkened glass plate upon which the

needle moves . . . An excellent plate darkener has been found to be a thin coat of vaseline covered with lampblack . . . Then there is the ordinary apparatus for projection . . . And last, but not least, there is the contrivance for determining the phase and amplitude of vibration, two elements in these figures only second in importance to time itself." For this purpose cords are drawn through the pendulum tubes. One of their ends is clamped near a knife edge, and their lengths adjusted here. From the lower ends of the pendulum tubes the cords pass through screw-eyes in the base, and then up to a board just below the top of the table, where three of the four are attached by hooks to pins which lock as many levers. To attach these hooks, the three pendulums must be pulled aside. When the fourth is then set swinging, its cord jerks the pin away from the third lever and thereby starts its pendulum, while the cord of this one performs the same service to the second, and this to the first.

It may be necessary to add that the two pen pendulums swing at right angles, while the two plate pendulums do likewise. If either the pen or the plate pair is stationary, the other pair draws a Lissajou curve. When all four pendulums are in motion, both X and Y have each two components. With their phases differing 90°, and their amplitudes equal, the pen as well as the plate draw circles. The amplitudes and the periods may then be adjusted for circular cycloids, the five-pointed star in 526B being given as an illustration.

Lissajou Forks

432. Under pendulum machines the old familiar Lissajou forks ought to be classed. They were a pair of large tuning forks, whose periods of vibration could be altered by clamping sliding weights to them. A small mirror was mounted on one prong of each of the forks, their planes of vibration being generally at right angles. A beam of light was reflected from one mirror to the other and then upon a screen where the rapidity of the vibration and the persistence of vision often showed the whole curve. While the relative periods and the amplitude were somewhat under control, the initial phases were a matter of chance.

Vibrating Rod

433. A vertical rod may be clamped to a firm support, and provided with a horizontal disk at the top. A pendulum with a tracing pen is then set swinging above the disk. This contrivance is, of course, a simple one, but in the hands of an expert it may give good results.

II. Wheel Machines

441. When a wheel rotates, the motion of its crank pin must be transferred along parallel lines in order to be truly simple harmonic. This is generally done by making the pin slide in a slot and move it in a direction at right angles to itself. The movement is then correct both in principle and in practice.

When the flywheel of an engine rotates with uniform speed, its piston cannot therefore have simple harmonic motion, because the pitman or connecting rod does not remain parallel to itself. The error may be corrected by a device due to Smedley, as given in "The Science of Musical Sounds" by D. C. Miller, page 11. Both of two ordinary and equal pitmen CA and CB in Fig. 441 are pivoted to the same

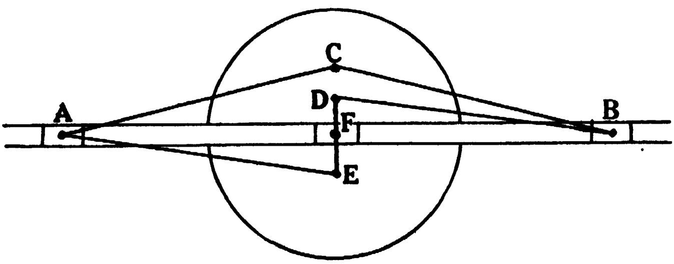

Fig. 441. Smedley's Method of Obtaining True Simple Harmonic Motion.

crank pin C and to opposite crossheads A and B. From each of these crossheads equal rods AE and BD run to a short bar ED pivoted to a block F, so that this slides with simple harmonic motion.

Another practical device is to lengthen the pitman, because the longer this is, the less does the motion of the piston differ from a truly harmonic one. The error may thus in practice be made negligibly small. This is the case in the Milne machine (445). But before taking this up, it will be well to dispose of a few simpler ones.

Pumphery's Cycloidotrope

442. This is a very simple and elegant little instrument well adapted for optical projection. A specimen of it is to be found in the physical cabinet of Creighton University. It was bought in 1883 for one pound ten shillings from J. H. Steward, of London. George M. Hopkins in his "Experimental Science," New York, Munn and Co., 1911, in vol. II, pp. 133-137, gives a description and picture of it. He ascribes its invention to A. Pumphery, of Birmingham, England.

Fig. 442 is a sketch of its principle. P is a needle point at the end of a rod PAC. C is a stud somewhat like an electric binding post. It is free to rotate on its support at the end of a crank EC, which may

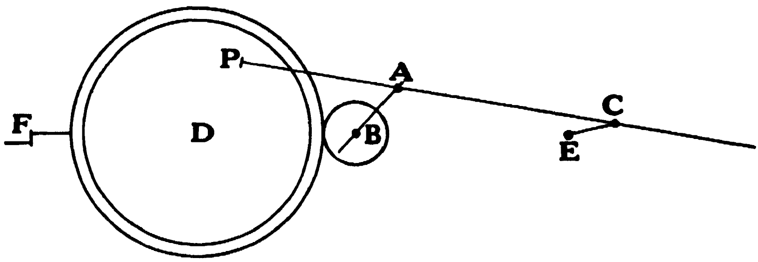

FIG. 442. Pumphery's Cycloidotrope.

be swung about E and clamped in any position. A is a similar stud at the end of a crank of variable length which may be clamped to the cog wheel B. This wheel B has 33 teeth and gears into the disk D which has 202 teeth. This disk is in reality only a ring, like a washer. A crank and gearing at F serve to rotate D and B. A smoked glass disk is laid upon D and held in place by two springs. The rod PAC is then clamped either at A or at C, and the crank F turned. As P may be placed anywhere on the disk and the lengths PA or PC and AB are adjustable, a great variety of figures may be drawn. These are not mathematically precise, because the rod PC does not remain parallel to itself. But this does not detract from their beauty.

THE RITCHIE MACHINE

443. In the physical cabinet of the Creighton University is an instrument in which the tuning forks in the ordinary LISSAJOU apparatus (432) are replaced by disks cut into circular sine curves with periods of 1, 2, 3, 4, 5, (like the larger curve in Fig. 625C, which has the period 8), and a duplicate of one of them, thus furnishing 10 combinations. Levers attached to two mirrors rest on these curves, and thus cause the mirrors to oscillate, one in a vertical, and the other in a horizontal plane. A beam of light reflected in the usual way from one mirror to the other, then shows the figures on a screen when the disks are revolved. The most elegant part of the machine is that the phase difference may be altered while it is running. This is done by having both disks mounted on a double shaft, one within the other, the inner one carrying a stud which slides in a spiral slot in the outer one. As one of these shafts is pressed outward by a spring against a

set screw, the turning of this screw by a crank causes it to rotate slightly on the other. When the machine is set in rapid rotation, persistence of vision will often show the entire curve, which may then be dephased at pleasure and all its varieties seen (713, 717).

The instrument bears no maker's name. As it was bought over 30 years ago when the entire cabinet, except its electrical and optical sections, was purchased from E. S. RITCHIE AND SONS, Boston, it was judged appropriate to call it the RITCHIE machine.

THE MORITZ MACHINE

444. This is a simple mechanical contrivance for drawing polar curves with the pen moving with one component in the radial line. It was designed and is described by R. E. MORITZ, University of Washington, Seattle, Wash., in the *Scientific American Supplement* of Aug. 5, 1916, No. 2118, pp. 84, 85. Fig. 444 gives both a vertical and a horizontal plan, only the most essential parts being represented or suggested, as the rest may be readily supplied.

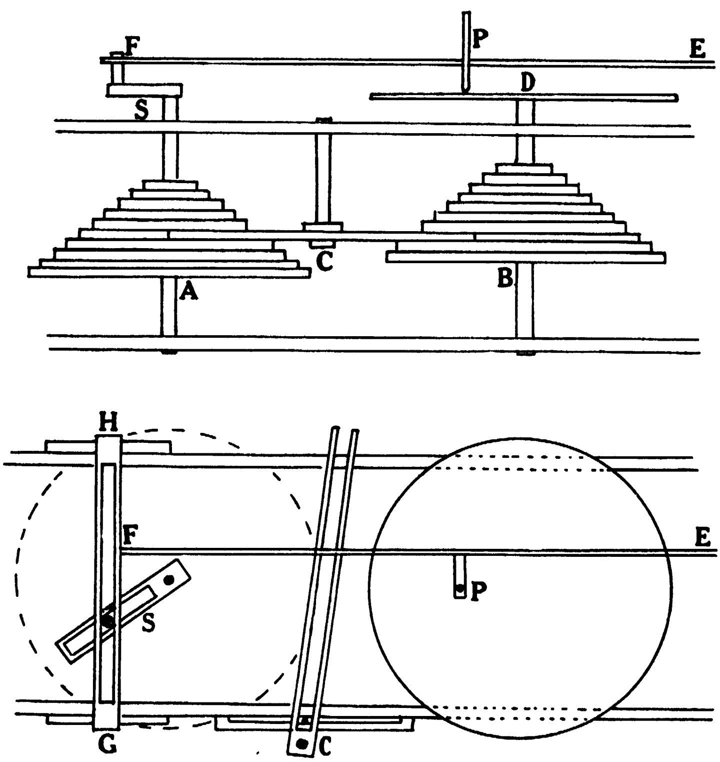

FIG. 444. The Moritz Machine.

A and *B* are two equal nests of 10 cog wheels each of various sizes. Each nest has all its wheels firmly fastened together. The entire set however may be clamped at any height on its shaft, so as to bring any one wheel of *A* into gear with any other of *B* through the intermediary idle wheel *C*, which may be swung round for this purpose. There are thus 63 possible combinations. *D* is the disk on which the paper is fastened, and *P* the pen. This last is attached to a rod *EF*. The *E* end slides through a support, while *F* is fastened to a slotted bar *GH*, in which the pin of the crank *S* slides. Various scales on the crank, the pen bar, and the wheel nests facilitate adjusting the instrumental constants. All the 63 curves that the machine can draw are illustrated in a publication of the University of Washington, Vol. I, No. 2, June, 1923, and their polar and Cartesian equations given. These curves are subdivided into their curtate, cuspidal, prolate, and equifoliate varieties (342), so that there are 252 different figures.

THE MILNE MACHINE

445. This machine is described in the article "A New Form of Harmonic Synthetiser" by J. R. MILNE, D. SC., in *Modern Instruments and*

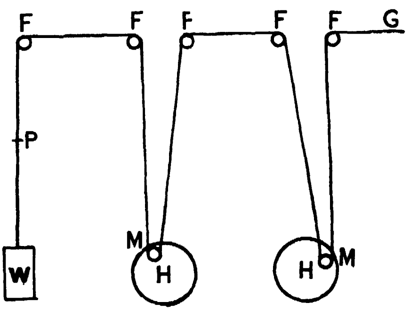

FIG. 445. The Milne Machine.

Methods of Calculation: A Handbook of the Napier Tercentenary Exhibition in 1917, pp. 337-339. This machine draws compound sine curves, like the Kelvin and the U. S. C. and G. S. tide predictors. Pulleys are put on the crank pins *MM* in Fig. 445 of the harmonic wheels *HH*, and a wire, anchored at *G*, run over them and over the idle *F* pulleys to the pen *P* and the weight *W*. The distances *FM* are about 30 inches. The amplitudes *MH* are not given. They are probably less than an inch, so that "the deviation from true simple harmonic motion, due to the finite length of *FM*, is insensible."

"The intention was to use the apparatus to draw a large number of different curves of known harmonic constituents to serve as standards of comparison. It was hoped that in this way the general species of a limnograph curve might be recognized by inspection, thus saving much exploratory calculation" in the analysis of an unknown compound sine curve.

The chief merit of the machine, however, consists in this that "it is possible to alter gradually the period and amplitude of the constituent harmonics *while the machine is in motion* . . . In order that the periodic time of the harmonic wheels may be variable at will, it suffices to connect each to the motor through the intermediary of two coned pulleys; the belt connecting each pair of pulleys can then be slid along them by means of a suitable guide, so as to alter the gear of its harmonic wheel to the motor as desired."

In order to vary the amplitude "the harmonic wheel is made in duplicate, the two wheels being connected by a crown wheel so as to form the differential gear now familiar to every one because of its use on motor cars. The effect of this duplication of the harmonic wheel is to set up *two* simple harmonic motions in the wire of equal period and amplitude, but differing in phase by an amount which depends on the position of the crown wheel. By displacing the axle of the latter through 90° the phase difference of the harmonics can be altered from 0° to 180°."

The Dechevrens Campylograph

451. This machine takes its name from καμπυλος, curve, and γραφω, I write or draw. It was designed by Marc Dechevrens, S. J., Jersey, England, and was probably the first to draw Lissajou curves on a rotating disk, and thereby give the pen a triple motion. It was made in two forms.

The first, Fig. 451, carried a large disk with 14 concentric crown wheels, all with the same pitch. Four rods, at right angles to one another, ran radially over this large nest of crown wheels to four nests of smaller ones with five wheels each, each rod carrying two adjustable cog wheels which could be set to engage any two crown wheels, one on the large and one on a small nest. The connection had to be such, however, that the equal wheels in the corners had to be joined on two opposite rods that formed the same diameter, although this rule could be violated for complex curves. There were 979 possible combinations. Shafts then ran up from the four small nests of crown wheels to the top of a table, carrying at their upper ends short slotted

and adjustable bars, which could be clamped at right angles to the shafts. These bars had crank pins at their ends and determined the amplitudes and phases, diametrically opposite ones being equal and

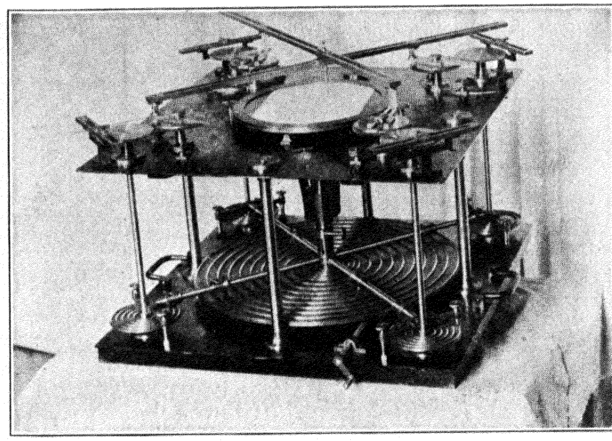

FIG. 451. The Dechevrens Machine. First Form.

parallel. Two long slotted bars, pivotted at each end to crank pins, crossed each other at right angles over the disk on which the paper was placed, and by means of sliders carried the pen at their intersection. A rod below the large nest of crown wheels, furnished with a crank and an endless screw, gave motion to the machine. This form of the machine was rather complex and expensive.

452. The second and improved shape of the campylograph is shown in Figs. 452. The lower picture presents the bottom or reverse side. The system of crown wheels is here replaced by a disk with equidistant small holes arranged in concentric circles, into which cog wheels gear with needle-like teeth. There are now only two sets of small wheels instead of four.

In the upper view, which shows the obverse or top side, there are two small graduated circles over which adjustable cranks move and determine the phases and amplitudes. The two crank pins give motion to two slotted bars which are at right angles to one another. These bars carry pivots, each being attached to one of two long parallel rods that run over the paper-covered and graduated disk, and carry the pen. As this pen must always be exactly midway, or at a given proportionate distance from the ends of the long rods, these last slide past one another and thus adjust their lengths. Near the pen they are furnished with racks, and in this way rotate the pinion in which the pen is cen-

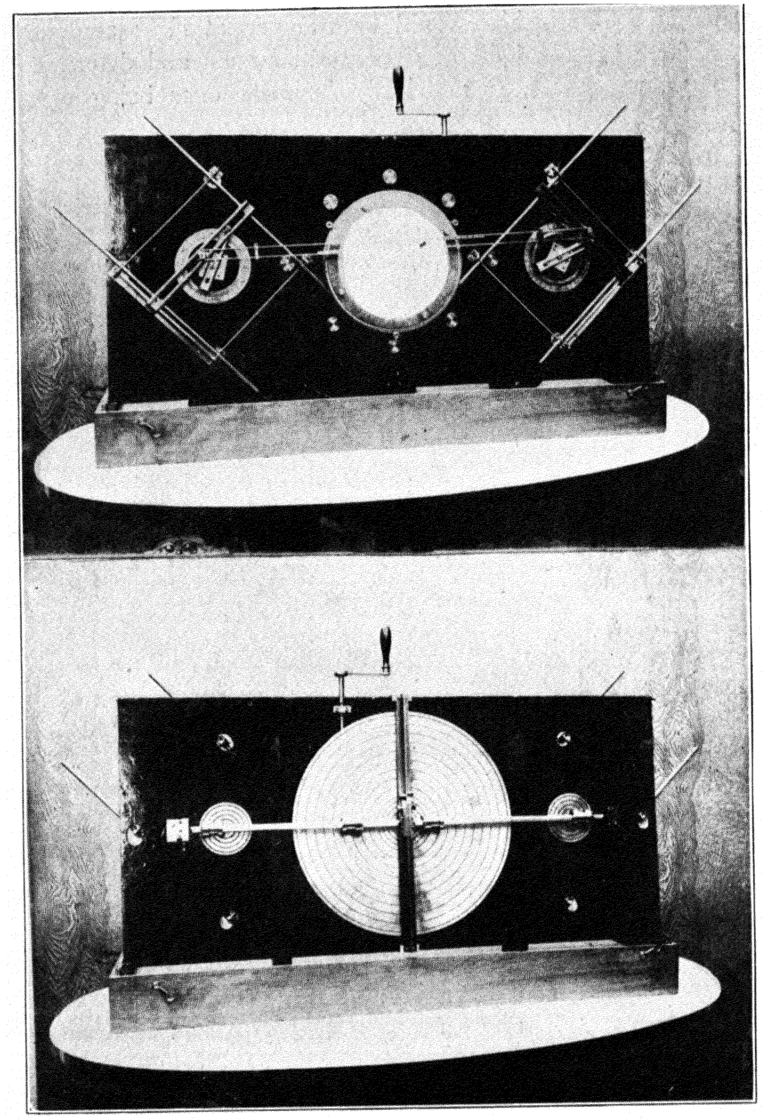

FIG. 452. The Dechevrens Machine. Improved Form.

tered, and keep it in its proper place. Motion is given to the machine by a crank.

453. Pere DECHEVRENS made beauty his first object. He certainly achieved his purpose, as is attested by the wonderful grace and variety of his drawings. He gave the writer of these lines certain rules for this purpose, which will be found in Chapter VIII on Beauty. The

mathematics of his figures, which embrace only one Y component, or only one in X and one in Y, over a stationary or rotating disk, together with simple sine curves, for which some extraneous mechanism had to be attached, was written by P. POTRON, S. J., and published in a pamphlet of 16 pages in 1902. While applying it to his various classes of curves, he gives no practical examples and no illustrations. The mathematics in this book was based on his, and then extended to many X and Y components and to rectangular-sine and to stereoscopic figures. The campylograph itself was presented to the French Academy of Sciences on June 11 and August 13, 1900. *La Nature* and *Cosmos* had several articles about it. A detailed description appeared in January, 1901, in the *Revue des Questions Scientifiques de Bruxelles.* None of the above have been accessible to the writer except the pamphlet mentioned before. He has however carried on a lively correspondence with Pere DECHEVRENS, and received a large collection of figures from him. DECHEVRENS has had for several years an elaborate work with numerous illustrations ready for the printer, but as he died on December 6, 1923, it will very likely never be printed.

THE CREIGHTON COMPOUND HARMONIC MOTION MACHINE

461. This machine* (see Frontispiece 461) was built by the writer in sections as the ideas advanced. It was not at the start conceived in its present entirety, so that, if he were to design another, the experience acquired would suggest several improvements. It was begun in January, 1915, and completely finished only in May, 1924.

Mathematical accuracy in principle was made the first essential of the machine. This proved to be a very fortunate decision, not only because the nature of curves could thus be thoroughly examined and the way paved for Chapter II of this book, but also because this made it possible to discover new methods and properties, as may be seen in five articles on the subject† and in many parts of this work.

462. The vital principle of the mechanism is shown best in Fig. 463. From the center of each of three graduated circles an axle protrudes and carries an adjustable crank pin at its end. This pin is screwed into a square rod sliding in a square tube, both ends of the rod being rounded and threaded so that by means of opposing nuts the crank

* This machine was described and illustrated in an article with the above title in the *Scientific American Supplement,* 1918, Feb. 9 and 16, pp. 88-91, 108 110.

† These are the four in the Appendix, and the one on Stereoscopic Harmonic Curves in *School Science and Mathematics,* xxiv, 29, January, 1924.

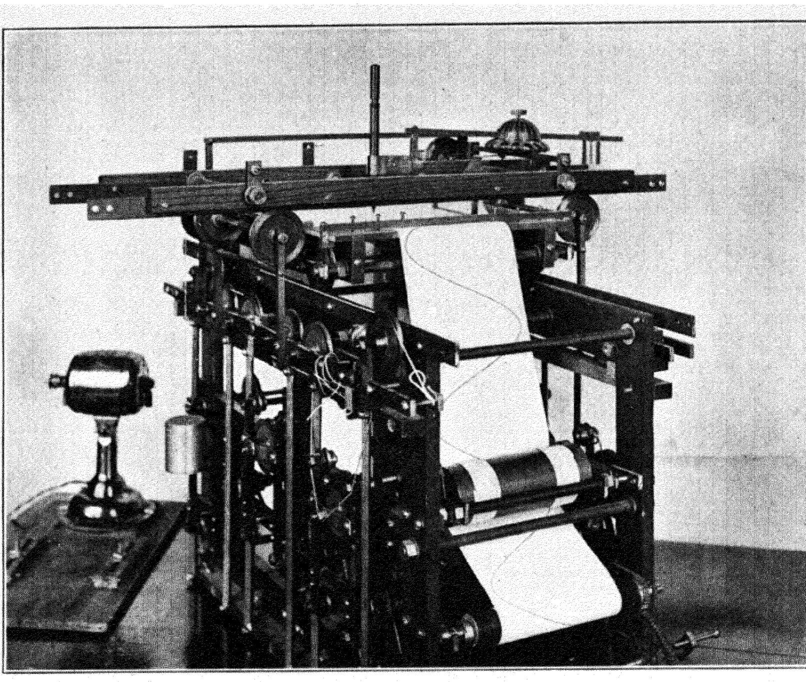

FIG. 462. The Creighton Machine. Roller End.

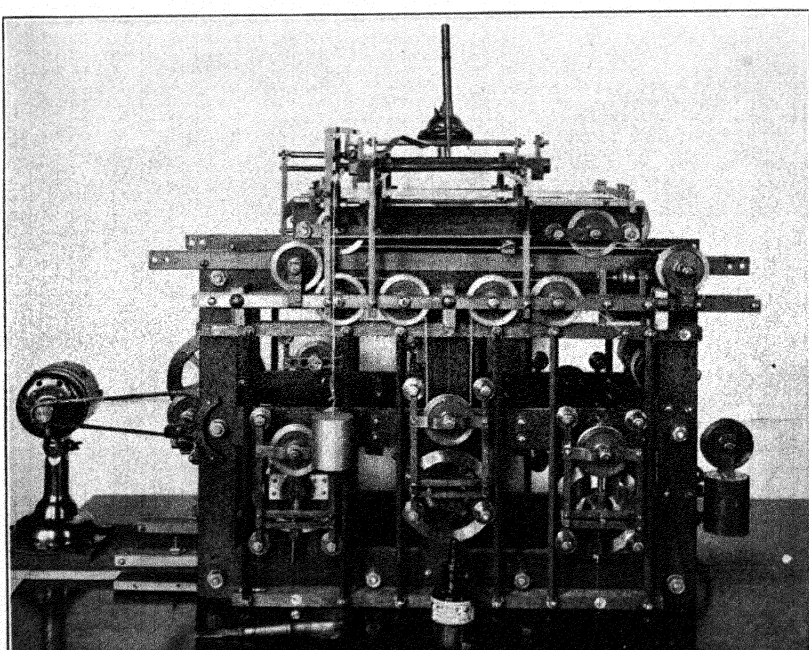

FIG. 463. The Creighton Machine. X Side.

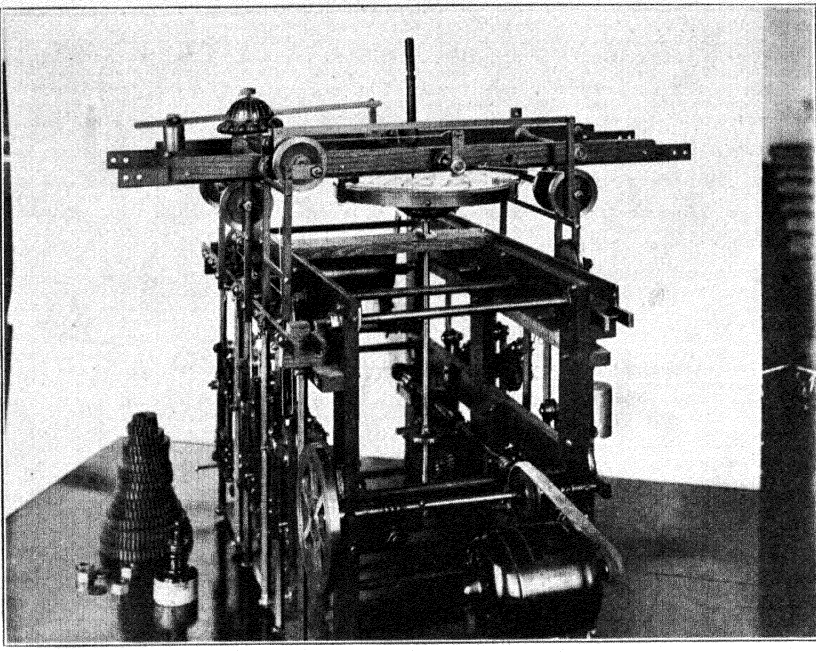

FIG. 464. The Creighton Machine. Motor End.

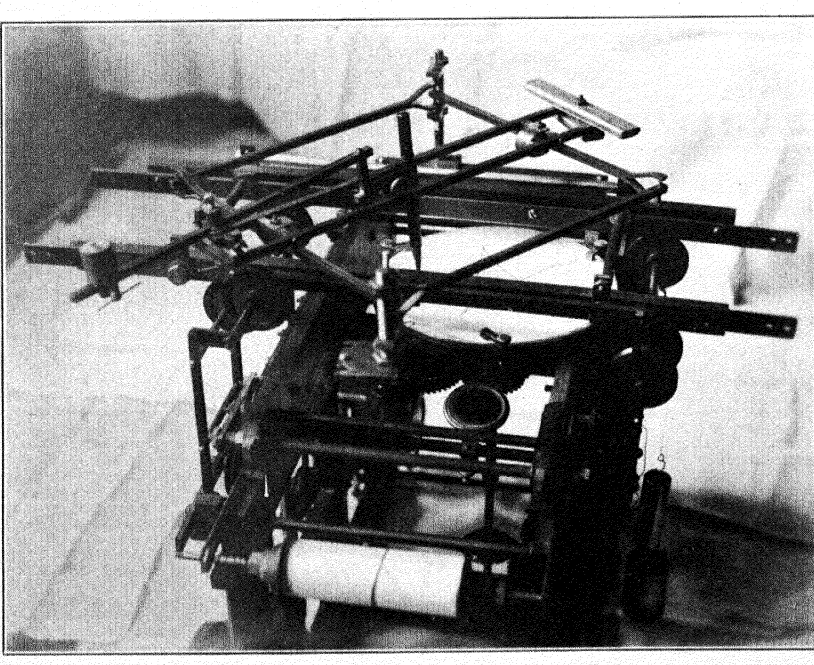

FIG. 465. The Creighton Machine. The Rhombus.

pin may be set at any distance from the center. As the axle revolves, the crank pin moves in a circle, and by sliding in a slot between two square bars, it raises and lowers a rectangular frame, which is held to a vertical course by means of four grooved rollers bearing against two parallel rails. A cord passes over the large grooved pulley in the upper part of the rectangular frame. One end of the cord is anchored to a firm support, while the other gives motion to a carriage holding a drawing pen or a sheet of paper against the opposing pull of a weight.

463. The Frontispiece 461 gives a broadside view of the machine, of what will be called the *Y* side. The strong oak framework, ½ inch thick, is 20 inches long and 14 inches high, and the sides are 8 inches apart. Like the other or *X* side, shown in Fig. 463, it presents three sections which are alike in all details. The sections are lettered, *A* being the right one in Fig. 463, *B* the middle, and *C* the left one. Similarly in 461, *D* is the right one. *E* the middle, and *F* the left one. A fine steel wire, No. 27 gage, with a thickness of 0.014 inch, is anchored above the *F* section, and may be run through any one or more of the *D, E, F* sections, then up to the *Y* carriage, from which another wire proceeds to the *Y* weight shown near section *C*. In like manner another wire is fastened in section *A*, and may combine any one or more of the *A, B, C* sections with the *X* carriage, whose weight hangs down to the right of *A* in Fig. 463. Small pulleys may be seen above the *A* and *F* sections, over which combination may be effected between the *X* and *Y* sides, and 4, 5, or 6 sections joined in series.

The fine steel wire, just mentioned, gave every satisfaction but one, and this was that after a long service it would break without warning, and thus at times ruin a fine curve that was nearly complete. A stout fishing line is now used and has proved to be perfectly reliable.

In Figs. 463 and 464 a small motor may be observed. Its office is to rotate a shaft which has a flywheel on its left or *Y* side. This shaft carries two worms or endless screws which gear into two long shafts, the right or *X* shaft being plainly in view in Fig. 464. These long *X* and *Y* shafts carry three worms, one for each section on their side, which gear into cog-wheels clamped on the inner ends of the axles.

464. These cog-wheels are ordinary commercial ones. They were intended by their makers, the W. F. and John Barnes Co. of Rockford, Ill., to serve as change gears on their smallest screw-cutting lathe, their 4½-inch, as they call it. They are all of the same pattern, of

iron, ⅜ inch thick, with a hole 19/32 inch in diameter, and of 32 pitch. This last expression means that a wheel with 32 cogs has a one-inch pitch radius, a 16-cog wheel has a radius of 16/32 or half an inch, a 48-cog wheel has a radius of 48/32 or 1.5 inch, and so on in proportion. The pitch radius is the acting one, so that the centers of two wheels must be placed exactly the sum of their pitch radii apart in order to gear properly. This is of special importance in the reduction gear of the disk, in which two pairs of such wheels are used. This may be glimpsed in Fig. 465. It was added after Fig. 464 had been made. And as the number of the cogs is proportional to the pitch radius, it is only necessary to select such pairs of gears as have the sums of their cogs equal. This is done quickly by means of a prepared table.

To allow for the variable size of the wheels, the journals in which they rotate and which carry the dials on their outer ends, may be raised and lowered in a slot and clamped on their inner ends, so that the wheels gear properly into the worms on the long X and Y shafts.

The gear wheels on hand have the number of their cogs multiples of 4, from 16 to 64. There are also wheels with 30, 45, and 50 cogs, so that there are 16 different wheels. Twelve of these have one duplicate, and three have respectively 2, 3, and 4 duplicates. The stock and the variety may be increased at any time. The assortment of wheels carried by the makers is such that every ratio from 1 : 1 to 1 : 100 may be had.

465. The Creighton machine will in principle draw any form of harmonic curve, for the reason that the number of its sections and the ratios of its gears may be made almost anything. It is the first machine, as far as is known, to use many components in series in both X and Y on a stationary and on a rotating disk, and on a moving ribbon.

466. Fig. 462 shows the machine arranged for sine curves with from one to six components in Y. The X carriage is held stationary. A ribbon of paper, 4¾ inches wide, the roll of which is visible in Fig. 465, is drawn over the plate glass top of the X carriage between two rollers in Fig. 462, which have a radius of one inch. The farther roller, which is hidden by the paper, carries a gear that meshes directly into the worm of section A, or by means of a reduction gear into section F. The largest ratio thus available between the cycles of the pen and those of the roller is 16 : 1, so that $y = b \sin 16x$ is the most "crowded" sine curve (212) that can be drawn in this way. By means of a worm to

be mentioned later (474), this ratio may now be made as high as 1680:1, but this contrivance will be classed under Rectangular-Sine curves.

467. In Fig. 463 the machine is adjusted for rectangular curves. Both carriages move with simple or compound harmonic motion. The *Y* carriage carries the pen and the *X* carriage the paper, the plate glass top of the latter insuring smooth drawing. As the pulleys in the brass frames of the sections are movable ones and the crank pins may be set a whole inch from their centers, the size of the figures, when two components are used in each axis, may be as great as 8 inches.

In Fig. 464 the machine is set for polar curves with one or more *Y* components in a radial or non-radial line. The disk shaft is driven from section *B*, the gear of which may do double duty. A mitre gear gives a ratio 1:1 and a bevel gear 2:1.

468. The Frontispiece, and especially Fig. 465, give views of the rhombus, by means of which several components in both *X* and *Y*, which by themselves would draw rectangular curves, may be used on the stationary or rotating disk, or on the ribbon of paper drawn with uniform speed by the rollers in Fig. 462 over a small board fastened above the stationary disk. For this purpose the former *X* carriage with its plate glass top is removed, and a second and longer one put in its place. This second *X* carriage carries an *X* pillar seen prominently to the left in the Frontispiece 461. The *Y* carriage carries a similar *Y* pillar supported by a short bar on its motor side. On these pillars the rhombus is pivoted.

The rhombus in geometry is defined to be an equilateral parallelogram with oblique angles. A student would call it an oblique square. This is made of brass rods, a quarter of an inch square, and all its corners are pivoted so as to eliminate all lost motion. The sides are 10 inches long. When the pen is placed in the line of both of the diagonals, it will remain at their intersection, no matter what the angles of the rhombus may be or the lengths of the diagonals. The essential point is then that the pen will always remain exactly halfway, or at any set proportionate distance, between the *X* and *Y* pillars. It will move parallel to both over half the distance that they move, or over a proportionate one. This division by 2 then counteracts its multiplication by the movable pulleys in the sections (467).

The pen is slightly underbalanced by a sliding weight on a lever, so that it touches the paper as gently as one may desire. When it is to be held away from the paper, a tooth pick is placed under its lever

and over the close parallel bars. This pen lever is then balanced by a counterpoise on the opposite side of the rhombus.

The pen is an ordinary stylographic pen using India ink. It has every desirable good trait, except that its tracings are rather thick. But even this one objection disappears when the figure is reduced by photography.

Fig. 465 also shows a small round mirror almost under the disk. This enables one to read the graduation of the disk to single degrees or to tenths of them. For this purpose the numbers were written reversed. The phases of the cranks in the sections may also be read and set to single degrees or tenths by means of needle points placed under them. The revolutions of the disk are registered by a Veeder counter, which may be seen just above the little round mirror. This counter is very useful when the disk makes many turns, so that one may know when the curve is nearly finished. No harm is done by letting the pen overrun its cycles, because, as the curves are "closed," it will merely retrace its path. This is however not only useless, but inconvenient when the initial phase is to be changed.

469. As the CREIGHTON machine was designed for harmonic curves of all kinds in general, but not especially for sine curves of the Fourier order (813), its three or six components in series are more than sufficient for ordinary purposes. Indeed, two components to each X and Y over a rotating disk or a moving ribbon give very complicated figures. And when beauty is the prime object, one component in each axis will be all that is needed. While the writer has actually tried all six components on a sine curve, he has never used more than two each on X and on Y simultaneously. The objections to more than two components on each axis are that the sum of their amplitudes becomes inconveniently large, the compound period is too long, and the lines overlap too much to be distinguishable.

THE NUMBER OF CURVES

The number of the various curves that may be drawn with the CREIGHTON machine with its 16 different wheels, when duplicates are used only in the reducing gear, is incredibly large. The method of computing this number will be given in brief, so that the reader may verify it if he is so inclined.

471. *Sine Curves,* in which the pen has components only in the Y axis, and the paper is drawn along the X axis According to algebra,

when n = total number of wheels and m = the number used at a time, there are $n!/(m!(n-m)!)$ combinations, in which $n!$ means factorial n, the product of all the successive numbers from 1 to n included. Thus, if 6 wheels are used at a time, there are $16!/(6!\,10!) = 8{,}008$ combinations. For 5 at a time there are 4,368; for 4, 1,820; for 3, 560; for 2, 120, and for 1, 16. Adding these together there are 14,892 different combinations.

Then the paper may be drawn along in many ways. When only one wheel is used on the roller in section *A*, there are 14 speeds, because the two smallest with 16 and 20 cogs are not attachable for this purpose. When the reducing gear is used with three wheels in section *F*, there are 83 ratios employing two wheels, which, with each one of the 14 remaining wheels meshing into the worm on the *Y* shaft, makes $83 \times 14 = 1162$ combinations, and these with the 14 mentioned before, give 1176 different speeds to the paper, and $14{,}892 \times 1176 = 17{,}512{,}992$ sine curves.

472. *Rectangular Curves*, in which there are one or more components in each *X* and *Y*. When *Y* has 5, there are 4368 combinations as seen before, and these multiplied by the 11 remaining wheels, of which one is used at a time in *X*, give 48,048 curves.

When *Y* has 4 components they may be arranged in 1,820 ways. When *X* has one, there are 12 times as many, or 21,840, and when its has 2 there are 66 times as many, or 120.120, and in all 141,960.

When there are 3 components in *Y* and 3 in *X*, there are $560 \times 286 = 160{,}160$. With 2 in *X*, $560 \times 78 = 43{,}680$, and with 1 in *X*, $560 \times 13 = 7{,}280$. In all 211,120.

When *Y* has 2 components and *X* 4, there are $120 \times 1{,}001 = 120{,}120$. With 3 in *X*, $120 \times 364 = 43{,}680$; with 2 in *X*, $120 \times 91 = 10{,}920$; and with 1 in *X*, $120 \times 14 = 1{,}680$. In all 176,400.

When *Y* has 1 component, there are only 16 combinations. When *X* has 5 there are 3,003 times as many, or 48,048. With 4 in *X*, 1.365 times, or 21,840; with 3 in *X*, 455 times, or 7,280; with 2 in *X*, 105 times, or 1,680; with 1 in *X*, 15 times, or 240. In all 79,098.

This makes a total of 656,626 rectangular curves.

473. *Sine-Polar Curves* differ from sine curves in that the paper is given a circular instead of a rectilinear motion. As there are 14,892 combinations for the *Y* axis, and as each of the 16 wheels may be put in section *B* to operate the disk, half of which may use the bevel ratio of 2 to 1, there are 357,408 curves. Then as there are 174 reduction

gears using four wheels, there are so many times more, or 62,188,992, and in all 62,546,400 sine-polar curves.

474. *Rectangular-Polar Curves.* In these the 656,626 rectangular curves may be drawn on the disk. As anyone of the 16 wheels may be placed in section *B* and half of them may use the bevel gear, there are 15,759,024 curves, in which no reduction gear is employed. With this there are 174 times, or 2,742,070,176 as many more, or in all 2,757,829,200 rectangular-polar curves.

A worm has lately been put on the disk shaft so that it can gear into any of the 16 wheels. By means of a mitre and an additional spur wheel this can make connection either directly with a gear wheel on the disk or indirectly through its reducing gear. This furnishes $16 \times (83 + 174) \times 656{,}626 = 2{,}700{,}046{,}112$ more curves, making in all 5,457,875,312 rectangular-polar curves.

475. *Rectangular-Sine Curves.* In these the paper may be drawn from under the 656,626 rectangular curves, first, with the 1,176 speeds mentioned for sine curves, thus giving 772,192,176 curves. Then by means of the worm on the disk shaft, which can use any one of the 16 wheels, and an intermediate idle spur and any one of the 83 two-wheel reductions and the bevel gear, there are $1\frac{1}{2} \times 16 \times 83 = 1{,}992$ additional speeds to the ribbon, thus adding 1,307,998,992 curves, making in all 2,080,191,168 rectangular-sine curves.

476. Adding up:

Sine	17,512,992
Rectangular	656,626
Sine-Polar	62,546,400
Rectangular-Polar	5,457,875,312
Rectangular-Sine	2,080,191,168
Grand Total	7,618,782,498

477. While this number is a little too large because a few ratios, such as 2: 1, may be obtained in several ways, which were however carefully excluded in the reduction gears, it is also a little too small because the bevel ratio excludes only 6, instead of 8, out of the 16 wheels, so that, while the precise number would be only a mathematical curiosity, on the whole it gives a correct idea of what may be done with 16 different wheels when duplicates are admitted only in the reduction gears. If the duplicates that are actually on hand were used in the combinations,

the total number would be much larger. Then the reduction gear, as its name implies, is used only to reduce the speeds of the disk and of the roller, but never to amplify them. And lastly, account was taken only of the periods, and none whatever of the amplitudes, initial phases, and positions of the pen on the disk, any one of which changes the appearance of a curve essentially.

478. At present the highest ratio of the periods of the pen to those of the disk is 1,680 : 1, and the lowest 1 : 4. In some cases it would be desirable to have the disk turn oftener than 4 times in one period of the pen, which is possible only with the largest wheel with 64 cogs and the smallest with 16. As the speed of the disk cannot be increased by reversing the reduction gear on account of the friction, this might be accomplished by attaching a reduction gear to the worm connection between the flywheel shaft and the long *X* and *Y* shafts. There is no urgent need for this, however, because when beauty is the chief object, a low ratio will never give it.

THE MOST COMPLEX CURVE

479. The most complex reduction gear on the disk consists of 64 : 52 and 56 : 60 (the sum of the cogs in each pair being 116), or of 224 : 195. With the bevel gear this is 448 : 195. Without using duplicates, the next best wheels to select are those with 50, 45, 44, 32, 28 cogs, with respectively 11,088, 12,320, 12,600, 16,325, 19,800 cycles when all are used on *Y* only or on both *X* and *Y*. A sixth wheel would not add to the complexity, except the one with 52 cogs, which is already employed, and which as a fact has no actual duplicate. As the wheel in *B* must make 448 turns during the 195 that the disk makes, the 50-cog gear when placed here would impose $50 \times 448 = 22{,}400$ revolutions on the *X* shaft, and this 16 times as many, or 358,400, on the flywheel. The wheel in *B*, however, must make 11,088 turns in one rectangular cycle, which is 24.75 times 448. As fractions cannot be used, the 24.75 must be multiplied by 4 to make it integral ($=99$), so that the 195 turns of the disk and the 358,400 revolutions of the flywheel must be multiplied by 99, giving 19,305 turns of the disk and 35,481,600 of the flywheel. As this generally makes 1,800 revolutions a minute or 108,000 in an hour, it would take 328.5 hours or 13 days 16.5 hours to draw this curve, that is, very nearly a fortnight. If instead of the 50-cog wheel any other were placed in *B*, the revolutions of the flywheel and the time would remain the same, but the turns of the disk would be greater than those given. And lastly, if in order to increase this com-

plexity, the worm gear on the disk shaft is used on the highest available gear with 64 cogs, the revolutions of the fly wheel must be multiplied by 64, so that it would take about 120 weeks or 2 years and 3 months to complete the curve. If the average time to draw a curve is then put down as one year, or even as one hour, the reader may figure out for himself how long it would take to draw all the curves that are possible with these 16 wheels, and how many more curves would be added to the list by the purchase of only one additional wheel with a different number of cogs.

III. The Advantages and Disadvantages of Pendulum and Wheel Machines

481. The advantages of a pendulum machine are

1. Its cheapness. Nothing can be cheaper and simpler than the DOBSON form.

2. The beauty of its curves, owing to the uniformly decreasing amplitude.

3. The very close ratio that may be given to the components.

4. The oblique angle that the X and Y components may make with each other in a rod machine.

482. On the other hand a pendulum machine has many disadvantages.

1. A wire pendulum is strictly confined to rectangular curves with only one X and one Y component. A rod pendulum may have at most two components in each axis.

2. As time is the most essential element, a pendulum must be permitted to swing from start to finish without the least interference.

3. For this same reason a pendulum cannot be used to describe sine and polar curves in which the paper is to be moved or rotated with a commensurate period, because this requires clockwork of a high order of precision.

4. A definite phase difference, unless it be 0° or 90°, as well as a definite period, are impossible to obtain except by trial.

5. The constantly decreasing amplitude, and the interference between pendulums mounted on the same base, unfit the pendulum machine for mathematical study.

491. The advantages of a wheel machine are many.

1. The periods, amplitudes, and initial phases of all the components may be determined with precision in advance.

2. As time is no essential element, friction and irregularities of speed are of no consequence, so that the machine may be stopped and the phases read at any point, and it may even be reversed.

3. There is every certainty that the periods, amplitudes, and initial phases of all the components will remain constant.

4. For this reason a wheel machine that uses cogs is pre-eminently adapted to the mathematical study of curves.

5. A machine may be built very rugged, and the stylus made to engrave the curve in a hard material. The pen may be lifted off the paper and set down again without fear of derangement, so that parts only of curves may be drawn.

6. A machine employs the geometrical principle: it emphasizes the *position* of the tracing point, and not its speed. The three elements of harmonic motion, the period, amplitude, and phase of each component are therefore always in plain view.

7. The close ratios of the pendulum are also possible in a wheel machine, either by cogs or by belting, especially by coned pulleys, and by a disk drive.

8. A wheel machine may draw every kind of harmonic curve.

9. When a curve has been drawn, a second, and a third, and as many as may be desirable, and of every variety, may be placed on the same paper or on any part of it.

10. The curves may be drawn in India ink and thus given at once to the photo-engraver.

11. The wheel machine may also be readily used for optical projection.

492. The disadvantages of a wheel machine are

1. Its cost. Simple machines like those of PUMPHERY and RITCHIE are very cheap, but then they are confined to one class of curves. The MORITZ machine is also limited to one class. The DECHEVRENS and CREIGHTON machines are very costly. Skill in the use of tools and time may save very much money. The wheels, pulleys, and other material of the CREIGHTON machine cost about fifty dollars.

2. Mechanical imperfections. Unless the workmanship of the machine is the best, the wheels and pulleys true and well centered, the shafts perfectly straight, and the like, there will be very undesirable vibrations and deviations of the pen, very difficult to trace to their sources, and then often impossible to remedy. These mechanical defects are not, however, always an unmixed evil. As they are generally

periodic in character, they may cause pleasing interferences, and produce a moire or watered effect, such as may be seen in Fig. 942,* 932, 922, 924B, and in many others.

Summary of Chapter IV

There are two distinct classes of machines, one employing pendulums and the other wheels.

PENDULUM MACHINES

(411, 412) General principle.

The Dobson duplex pendulum (413) uses a wire. It is the simplest of its class.

The Hopkins pendulum (421) is also duplex, but uses rods.

The Hoferer quadruplex employs four rod pendulums (431), two for the pen and two for the plate.

The Lissajou forks (432) consist of a pair of large tuning forks, each with a small mirror on one prong.

WHEEL MACHINES

(441) General principle.

The Pumphery cycloidotrope (442) draws only circular cycloids or quasi cycloids. It is very simple.

The Ritchie machine (443) produces Lissajou curves by means of wheels instead of tuning forks.

The Moritz machine (444) draws sine-polar curves.

The Milne machine (445) draws compound sine curves, whose periods and amplitudes may be changed while it is in operation.

The Dechevrens campylograph (451-453) draws about a thousand curves. These are rectangular with one component only to the pen in each axis, while the paper is stationary or rotates.

The Creighton machine (461-479) draws every class of harmonic curves, sine, rectangular, sine-polar, rectangular-polar, and rectangular-sine. While only six components are available, there is no limit to their number in principle. The pen may have one or more components in one axis only or in both, while the paper is either stationary, or is moved with uniform rectilinear or rotary speed. The number of the curves that the Creighton machine can draw (471-479) is incredibly large.

THE ADVANTAGES AND DISADVANTAGES OF PENDULUM AND WHEEL MACHINES (481, 482, 491, 492).

CHAPTER V

CYCLOIDS

511. A cycloid (Greek κυκλος, circle), in the widest acceptation of the word, is a curve which is in some way derived from a circle. In this sense which is seldom employed, it would include the conics and all their derivatives, and very many other curves.

In a narrower, and more generally accepted meaning, a cycloid is a curve traced by a point on a rotating circle while the center of this circle revolves in a second one. The center of the second circle may also be made to revolve in a third circle, and so on indefinitely. In this case the definition embraces all harmonic curves (235).

In its most restricted sense, a cycloid, or *the* cycloid, or *the common* cycloid, is always taken to be the curve generated by a point on a circle as this rolls on a straight line. It is the path drawn in the air by a lump of mud on the tire of a wheel of a moving wagon.

It is best to make the common cycloid the starting point of the present investigation, and to derive the other cycloids from it by a synthetic process.

The Common Cycloid

512. This was just defined to be the locus of a point on a circle which rolls on a straight line. The tracing point on the circle, therefore, has two motions, one of rotation, and one of rectilinear translation. In the rotation there is the usual relation between its co-ordinates, while the translation merely increases one of them, generally x, uniformly. Hence if a is the radius, the equation of the circle is (252)

$$x = a \sin(\theta - 90^\circ)$$
$$y = a \sin \theta.$$

The starting phase ξ is here minus, because the rotation is inverse. The rolling then adds $a\theta$ to x. Before applying this, it is to be noted that in the cycloid the angle θ is always reckoned from the lowest point of the circle on $-Y$ ($\theta = BCP$ on Fig. 255) instead of from $+X$ as usual. This adds -90° to the initial phases. Then the origin is shifted from the center to the line, so that the equation finally becomes

$$x = a\theta + a \sin(\theta - 180^\circ)$$
$$y = a + a \sin(\theta - 90^\circ).$$

As $\sin(\theta - 180°) = -\sin(180° - \theta) = -\sin\theta$
and $\sin(\theta - 90°) = -\sin(90° - \theta) = -\cos\theta$,
the simplified parametric equation is

$$x = a(\theta - \sin\theta)$$
$$y = a(1 - \cos\theta).$$

From these the usual Cartesian equation may be derived (255).

513. When the rolling circle is allowed to slip or slide on the line, the law of this slipping or sliding must, of course, be known. When it is uniform, the equation of the cycloid becomes

$$x = g\,\theta - a\sin\theta$$
$$y = a(1 - \cos\theta).$$

In uniform slipping, as in the occasional spinning of locomotive drive wheels at starting, the peripheral velocity is greater than the translatory, and $a > g$, so that the tracing point is practically beyond the circumference of a circle which would roll on a parallel line without this slipping. The cycloid is then said to be prolate.

In sliding, as on a slippery track, the peripheral velocity is less than the translatory, and $a < g$, so that the point may be conceived to be within a circle that would roll without sliding. The cycloid is then curtate.

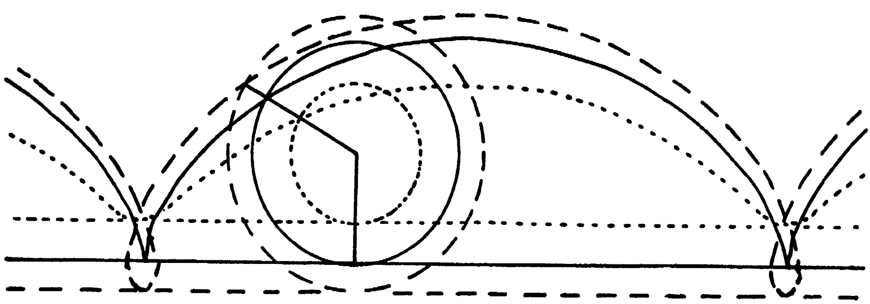

FIG. 513. The Common Cycloid, Prolate, Right, Curtate.

All three classes of cycloids are exemplified in Fig. 513. The full-line curve is the common and regular cycloid, drawn by a point on the full-line circle as this rolls on the full-line below it. The dashed curve is a prolate cycloid, which is drawn either by a point on the produced radius of the circle just mentioned, or by a point on a larger circle as this rolls and slips uniformly on its line. The dotted curve is a curtate cycloid, which is drawn either by a point within the first-mentioned

circle, or by a point on a smaller circle as this rolls and slides on its corresponding dotted line. At a given position of the generating circle, the angle θ is the same in all three curves.

514. Both classes, prolate and curtate, are sometimes called roulettes. When the rolling is firm, as that of a cogwheel on a rack, and as it is always taken to be, there is a cusp at each revolution of the rolling circle. In a prolate cycloid the cusp is replaced by a loop (Fig. 513), and in a curtate one it is cut off and smoothed and rounded like the worn-out point of a tool. The presence of a cusp is therefore a practical and sure sign that the rotary and translatory motions are correctly proportioned, and are in fact at that instant equal and opposite, because at a cusp the pen is momentarily at rest. When a disk is used, the pen at a cusp is compelled both by the rectangular combination of X and Y and by the rotation of the disk to describe the same circle in the same direction and with the same speed, so that it is at rest with reference to the paper. Indeed, it seems to have broken away from the rotating mechanism and dug itself into a hole in the disk so as to be carried away by it alone. When the equation of the curve is differentiated, the differentials of both co-ordinates will be found to be zero at a cusp.

Circular Cycloids

521. When the center of the rotating circle moves in a second and fixed circle, the tracing point describes a circular cycloid. This is generally called an epicycloid when the center of the rotating circle revolves in the same rotary direction as the tracing point, and a hypocycloid when the direction is not the same in both circles.*

A circular cycloid may be drawn mechanically in at least three ways. The first and most obvious way is to give a rotary motion to the pen and then rotate a disk under it, as is attempted in Pumphery's cycloidotrope (442).

522. A second method is by means of two pairs of rectangular components with respectively equal radii and at right angles in phase. This way may be used only on the Hoferer and the Creighton machines, because it requires two components in each axis.

523. A third method is a combination of the first and second, that is, by one pair of rectangular components and a rotating disk. This may be used on both the Dechevrens and Creighton machines.

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* Except perhaps when $m < 1$. (527 and 551.)

According to the definition of a circular cycloid (521), the center of one circle moves in the circumference of the other. The two circles cannot therefore have the same center (641-644). This contingency can never happen in the purely rectangular method (522), because here the co-ordinates are always added. It can happen only when a disk is used, and here only when the center of the rectangular circle is placed upon the center of the disk, the radii of the two circles being thus equal. In this case the fixed circle is a point, and the curve (or circle) drawn can be called a cycloid or a harmonic curve (233, 235) only when this point is taken as a circle with a zero radius. For this reason this method will be called a mechanical one, all the others as usual harmonic (641-644). In this one case it is the periods or angular speeds that are added, whereas it is the co-ordinates in all other methods. These facts may be used to advantage as artifices (641).

EQUATIONS

524. The second and third methods mentioned above suggest two ways of establishing the equation of a circular cycloid. That from the rectangular components is (223)

$$x = a \sin (m\,\theta \pm 90^\circ) + b \sin (n\,\theta \pm 90^\circ)$$
$$y = a \sin m\,\theta + b \sin n\,\theta.$$

and that from one rectangular pair and a rotating disk (237) when

$$\xi = -90^\circ, \quad x = -a \cos (m+1)\,\theta + r \cos \theta$$
$$y = a \sin (m+1)\,\theta - r \sin \theta.$$

and when $\xi = +90^\circ$, $x = a \cos (m-1)\,\theta + r \cos \theta$
$$y = a \sin (m-1)\,\theta - r \sin \theta.$$

In these equations, and in all those that are to follow, the first terms on the right side always refer to the moving or generating circle, the second to the one in which the center of the rolling circle revolves.* Unlike signs in these pairs denote an inverse or clockwise rotation, like signs a direct or anticlockwise one (223).

525. Quite generally a circular cycloid is defined to be the curve generated by a point on a circle which rolls on another circle. There must evidently be a fixed ratio between the radii of these two circles, when one is to roll on another in a given number of times in order to bring

* This rule however is not essential, as the center of either circle may revolve in the circumference of the other and generate the same cycloid (562, 563).

its tracing point back to its place of starting. As the radius of the fixed circle, m, on which the moving one rolls, differs from the radius, r, of the circle in which the latter's center revolves, by exactly the radius of the moving circle, a, (Fig. 528), it is more convenient mathematically to use the radius r instead of the radius m in the general equation. This r is then the same as in 237 or 524. The meaning of m will be seen later.

526. When $r = ma \pm a = a\,(m \pm 1)$, the curve generated is a true or cuspidal circular cycloid (514). When r is less than this, and $\xi = +90°$, it is a prolate epicycloid, and similarly when $\xi = -90°$, it is a prolate hypocycloid. When r is greater than the above value, the cycloids become curtate. Prolate cycloids of all kinds may always be diagnosed by their loops, curtate ones by rounded or blunted points, and true ones by cusps (514). These cycloids, however, become more and more difficult to identify the farther r is from the value $a\,(m \pm 1)$.

Fig. 526 shows all these classes of curves. The upper figures, A, B, C, are hypocycloids, the lower ones, D, E, F, epicycloids. The middle ones, B and E, are true or right; the left ones, A and D, prolate; and the right ones, C and F, curtate. The figures were all drawn to the same scale, m being equal to 5/2, that is, as will be seen later, 5 cusps (or loops or blunted points) in 2 revolutions of the rolling circle within or about the fixed one. The three upper curves, as well as the three lower ones, might have been superposed as in Fig. 513, and the same conclusion drawn. But this would have congested the diagram too much.

When the radius of the center-circle is $r = a\,(m+1)$ and $\xi = -90°$, the equation of a true *Epicycloid* is (238)

$$x = -a \cos (m+1)\,\theta + a\,(m+1) \cos \theta$$
$$y = a \sin (m+1)\,\theta - a\,(m+1) \sin \theta.$$

When $r = a\,(m-1)$ and $\xi = +90°$, the equation of a true *Hypocycloid* is (239)

$$x = a \cos (m-1)\,\theta + a\,(m-1) \cos \theta$$
$$y = a \sin (m-1)\,\theta - a\,(m-1) \sin \theta.$$

For an epicycloid the radius of the fixed circle on which the other rolls, is, as said before, $r - a = ma$, and for a hypocycloid $r + a = ma$. In most of the curves drawn, the radius a of the rolling circle was taken as the unit, one inch or 2.5 cm. in actual work, so that generally the radius of the fixed circle $= m$.

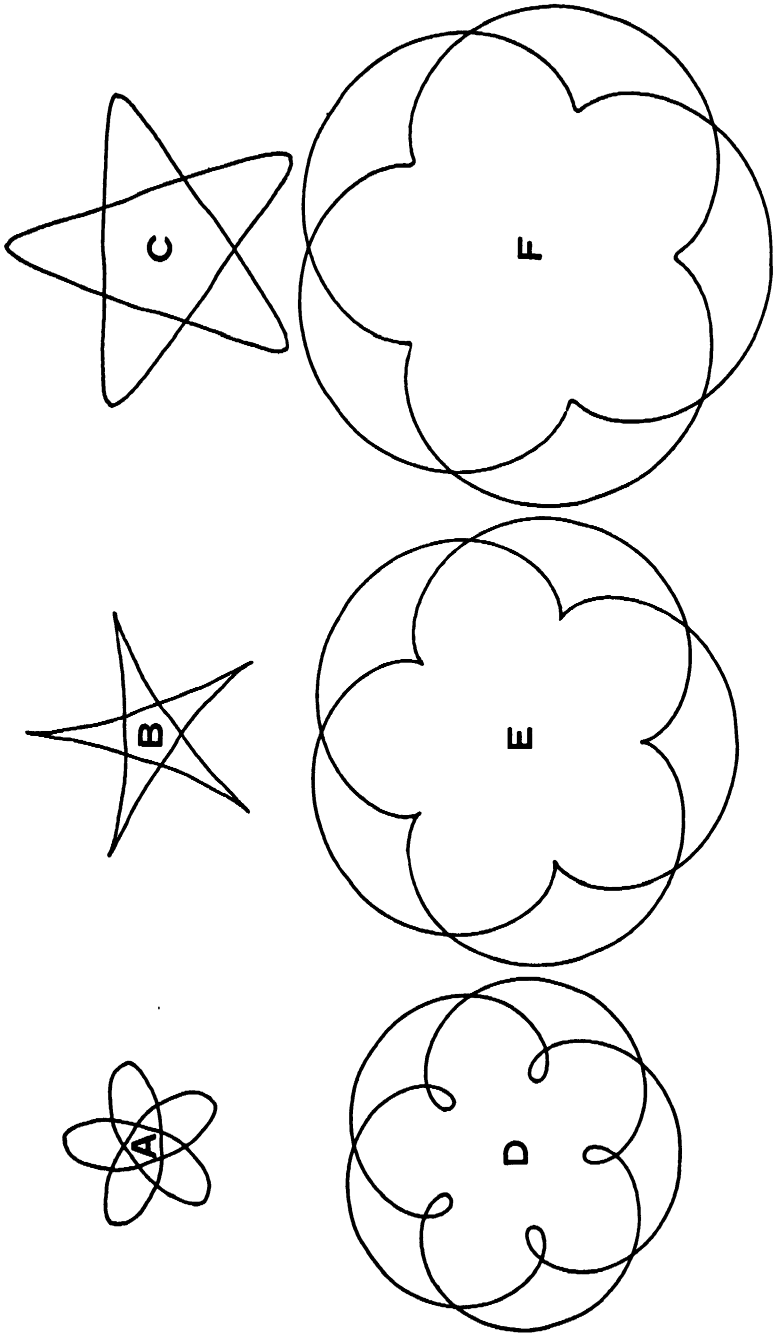

Fig. 526. Hypocycloids and Epicycloids, Prolate, Right, and Curtate.

527. In an epicycloid the signs of the terms remain the same for all values of m, because m is essentially positive. The inequality of the signs in the first and second terms then shows that the direction of rotation of the rolling circle and of its center are both inverse or clockwise (223 and Fig. 528A). And as $m+1>1$, (the coefficients of θ), the rotary speed of the rolling circle is always greater than that of its center (Fig. 528A). In a hypocycloid when $m>1$, the rolling circle rotates directly while its center revolves inversely (Fig. 528B), but when $m<1$, both rotate inversely as in the case of epicycloids (Fig. 528C), because the equation is

$$x=a\cos(1-m)\,\theta-a\,(1-m)\cos\theta$$
$$y=-a\sin(1-m)\,\theta+a\,(1-m)\sin\theta.$$

In this case the hypocycloid may truly be called an epicycloid. This will be discussed more in detail later on (551 seq. and 562). The rotary speed of the rolling circle is now less than that of its center because $1-m<1$, while it is greater when $m>1$, for then $m-1>1$.

528. By one rotation of the rolling circle is always meant the complete rotation of its center entirely around the center of the fixed circle, from any one position, say on $+X$, or east, or to the right, to the same position again. It is the same as one complete turn of the disk, when this is used. The angle θ in the equation is however always taken absolutely in the usual trigonometric way, so that values greater than 360° occur quite generally. It will also be necessary to remark, that, no matter how the curve may lie on the page, the origin is always taken at the center of the fixed circle or that of the center-circle, and that the positive end of the X axis passes through the first cusp, or first point of contact, at which the tracing begins. The angle θ is measured at the origin from the axis of X, and in the second terms of the equation indicates the rotation of the disk, or of the center-circle, or of the point of tangency of the rolling circle, but $(m+1)\,\theta$ and $(m-1)\,\theta$ are measured (absolutely) at the center of the rolling circle.

The tracing point always moves inversely, so that starting at $x=+m$, $y=0$, when θ is 0, it at once descends into the fourth quadrant where x is plus and y minus. The proof of this may be based on theory, by analyzing the equations when θ is small, and on fact, by taking a numerical example.

The diagrams A, B, C, in Fig. 528, illustrate all three classes of cycloids, A being the general one for an epicycloid, B for a hypocycloid when $m>1$, and C for a hypocycloid when $m<1$. In all three figures O is the origin and the center of the fixed circle as well as that of the

circle in which the center of the rolling circle moves, R is the center of the rolling circle when $ROX = \theta = 30°$, P the point it is tracing on the cycloid. In A, $m = 4$, in B also $m = 4$, and in C $m = 3/4$, the

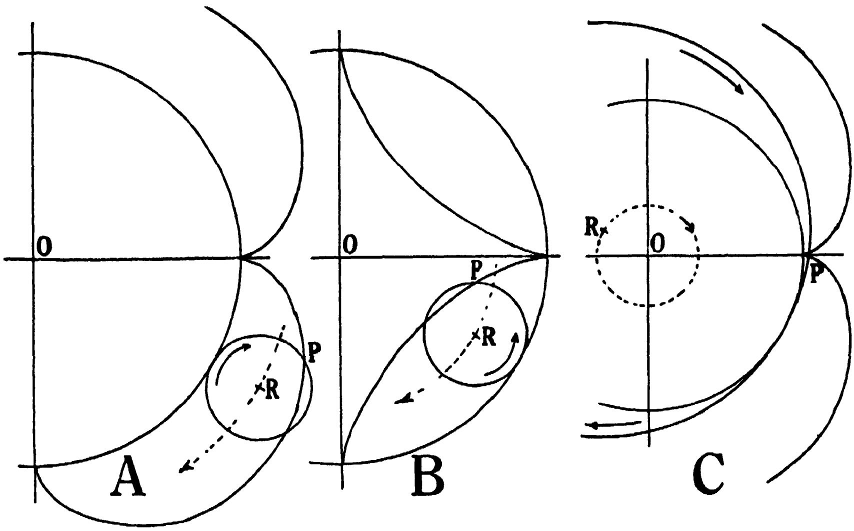

FIG. 528. Generation of a Circular Cycloid.

radius of the rolling circle being unity. The inclination of PR is $(m+1)\,\theta = 150°$ in A, $(m-1)\,\theta = 90°$ in B, and $(1-m)\,\theta = 7°.5$ in C, the zero points of these angles being as explained in (223).

Reciprocity

529. When the radius a of the rolling circle is habitually taken as the unit of measurement, it may be omitted from the equations, so that they become

for an *Epicycloid* $x = -\cos(m+1)\,\theta + (m+1)\cos\theta$
$y = \sin(m+1)\,\theta - (m+1)\sin\theta$

and for a *Hypocycloid* $x = \cos(m-1)\,\theta + (m-1)\cos\theta$
$y = \sin(m-1)\,\theta - (m-1)\sin\theta.$

The reciprocity between the coefficients of the terms and the coefficients of the angles is evident on mere inspection, so that it furnishes not only a ready test of the genuineness of an epicycloid and hypocycloid, but also a quick way of writing its equation.

When m is a fraction, it may be put equal to n/d = numerator/denominator. Then the above equations may be expressed directly in integers.

$$x = \mp d \cos (n \pm d)\,\theta + (n \pm d) \cos d\,\theta$$
$$y = d \sin (n \pm d)\,\theta - (n \pm d) \sin d\,\theta.$$

The upper signs are to be used for an epicycloid, the lower ones for a hypocycloid. Here $d\,\theta$ is an ordinary algebraic or numerical quantity, not a differential.

Epicycloids

531. An epicycloid, as ordinarily defined, is the curve generated by a point on a circle which rolls on the outside of another circle, so as to preserve external contact with it. The parametric equation, when the radius of the rolling circle a is unity,* is (529)

$$x = -\cos (m+1)\,\theta + (m+1) \cos \theta$$
$$y = \sin (m+1)\,\theta - (m+1) \sin \theta$$

and the polar $\rho^2 = 1 + (m+1)^2 - 2\,(m+1) \cos m\,\theta.$

The quantity m means the number of cusps per revolution. By the number of revolutions is here understood both the number of times the rolling circle runs around the fixed one (528), or the number of turns the disk makes in drawing the curve, as well as the number of radial intersections. The quantity m also means the radius of the fixed circle. The minimum radius vector ρ is likewise equal to m, and the maximum $m+2$. When m is a fraction it may be more convenient to replace it by n/d = numerator/denominator. The equation then becomes

$$x = -d \cos (n+d)\,\theta + (n+d) \cos d\,\theta$$
$$y = d \sin (n+d)\,\theta - (n+d) \sin d\,\theta.$$

Special Cases

532. *First* $m = 1$. Then

$$x = -\cos 2\,\theta + 2 \cos \theta$$
$$y = \sin 2\,\theta - 2 \sin \theta.$$

Shifting the origin to $+1, 0$,

$$x = -1 - \cos 2\,\theta + 2 \cos \theta$$
$$y = \sin 2\,\theta - 2 \sin \theta$$

from which $\rho = 2\,(1 - \cos \theta)$.

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*This will always be the case in this book unless the contrary is specially stated.

The epicycloid of one cusp is therefore a *cardioid.* No figure is here given. It would be identical with Fig. 551 except in size.

533. *Second* $m > 1$. Textbooks generally give the value of m in integers, such as $m = 4$ in Fig. 533A. Fig. *B* gives $m = 6/5 = 1.2$,

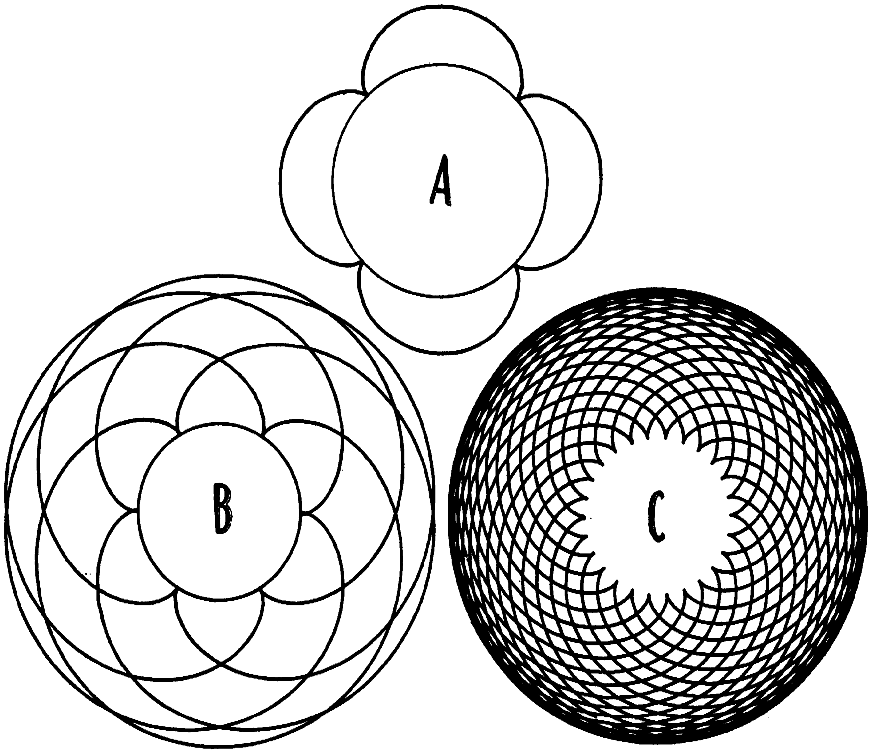

FIG. 533. Epicycloids $m > 1$.

that is, 6 cusps in 5 revolutions of the rolling circle about the fixed one. Fig. *C* shows $m = 26/25 = 1.04$.

534. *Third* $m < 1$.

Fig. A shows $m = 3/5 = 0.6$, B $m = 2/5 = 0.4$, C $m = 9/10 = 0.9$, D $m = 24/25 = 0.96$, E $m = 44/45 = 0.978$.

535. In all cases when the radius of the rolling circle a is taken as the unit, the radius of the fixed circle $= m$, and of the circle in which its center revolves $r = m + 1$, the minimum value of the radius vector ρ is m, and the maximum $m + 2$. These facts furnish a convenient

method of drawing with the CREIGHTON machine epicycloids of a whole series, in which, for example, d is a multiple of 5. But before taking an actual case, a few remarks about the practical method of proceeding will be in place.

The center of the disk is first found by placing the pen as near it as it may be estimated, and then, having loosened the set screw that holds the mitre or bevel gear to the shaft of the disk, the disk is given one turn by hand so that the pen describes a circle on it. The center of this circle, which is of course the center of the disk, is then found by placing the pen closer to it so that it draws a very small circle or better still remains stationary as the disk is revolved again, or two diameters at right angles are bisected by applying a graduated scale. Having found the center of the disk, a radial line is then drawn through it by moving the Y carriage, and the distance m from the center marked on $+Y$. The operator is supposed to look at the curve from the Y side of the machine (see Frontispiece) when there is question of the positive and negative directions of the axes X and Y. During this operation the X carriage is supposed to be connected in phase $0°$ with its weight attached, the disk and the Y carriage disconnected, and free. The Y carriage is now connected in phase $-90°$, the rhombus is adjusted so as to bring the pen to $Y=+m$, a new sheet of paper is placed on the disk, and lastly the disk also is connected and the motor started.

In Figs. 534 A to E a pair of 50-cog wheels was placed in X and Y, their phases adjusted at right angles, and their amplitudes set to one inch. These combinations remained the same throughout the series. For $m=3/5$ in Fig. A a 30-cog wheel was placed in section B so as to operate the disk and make it rotate $5/3$ as fast as X and Y. The pen was set accurately at the minimum value of $\varrho=m=3/5=0.6$ inch from the center, the disk clamped up and the motor started. For $m=2/5=0.4$, the pen was placed 0.4 inch from the center after having put a 20-cog wheel in B. A 45-cog wheel in B with the pen at $m=45/50=9/10=0.9$ inch from the center, gave Fig. C. A 48-gear with the pen at $m=48/50=24/25=0.96$ inch out of center gave Fig. D. Many other epicycloids not here illustrated were also drawn rapidly in this way.

536. In the cases last mentioned m was less than unity, and all the figures were of convenient dimensions. But when m is much greater than unity, the drawings become too large for the disk. This may be remedied by reducing the unit, as in Figs. 526 and 551, and taking it equal to half an inch or so. When m is slightly greater than 1, as in

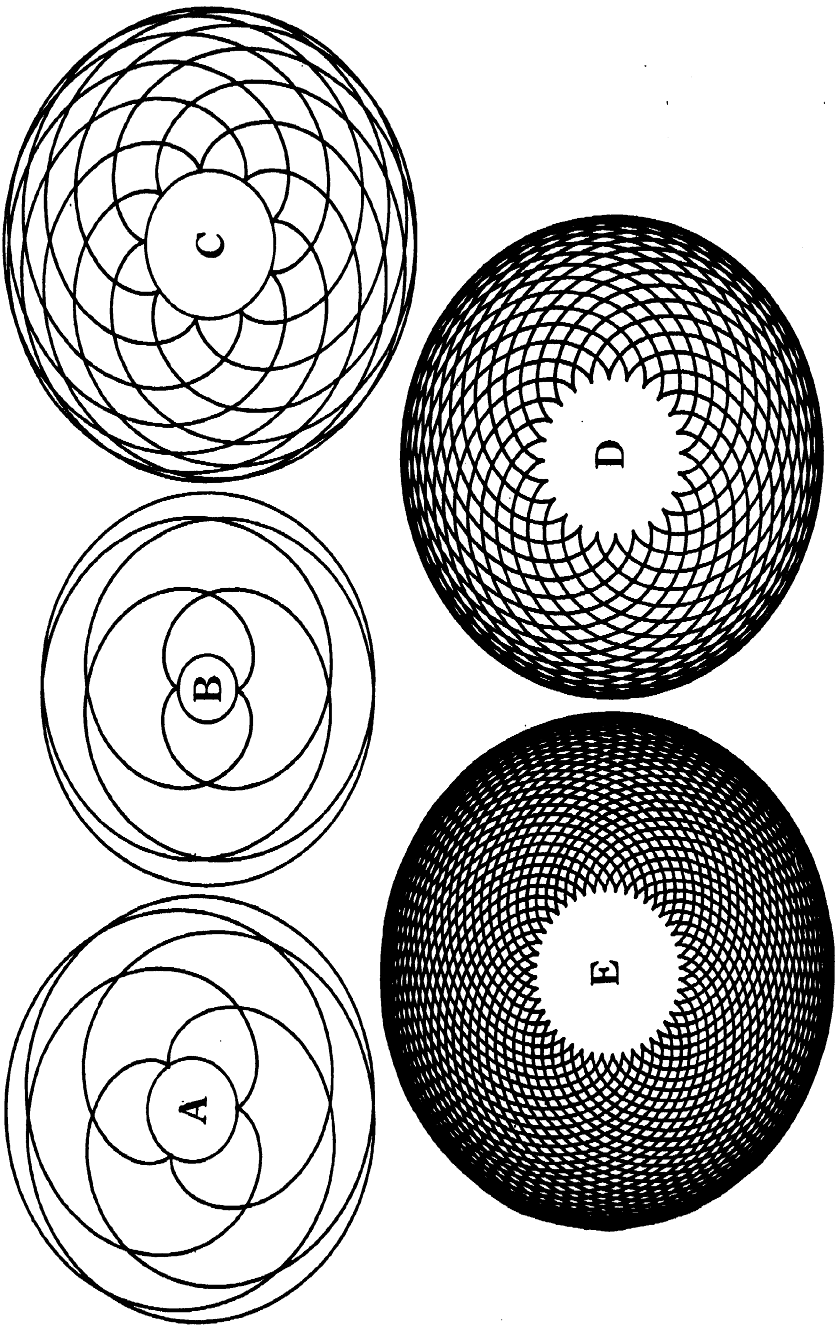

Fig. 534. Epicycloids, $m < 1$.

Fig. 533C, the figures do not differ in appearance from those in which it is slightly less, as in Fig. 534D. The cusps m are readily counted, but d, the number of points in which a radial line cuts the curve, and which are here indicative of the number of revolutions, become undecipherable when they are crowded together as in Fig. 534E. Even in Figs. 533C and 534D it would be hard to prove $d = 25$.

Hypocycloids

541. A hypocycloid is generally said to be the curve traced by a point on a circle which rolls on the inside of another circle. As this definition may cause trouble when the rolling circle is larger than the fixed one, it may be well to amend it by saying that a hypocycloid is traced by a point on a circle which rolls on another so as to preserve internal contact with it. The radius of the rolling circle a will as usual be taken as unity. The parametric equation is then (529)

$$x = \cos (m-1)\,\theta + (m-1) \cos \theta$$
$$y = \sin (m-1)\,\theta - (m-1) \sin \theta$$

and the polar $\rho^2 = 1 + (m-1)^2 + 2\,(m-1) \cos m\,\theta$.

The quantity m is, as before, equal to the number of cusps per revolution, to the radius of the fixed circle, and also algebraically to the maximum radius vector of the curve, $m-2$ being the minimum radius vector. When m is a fraction, it may be replaced by $n/d =$ numerator/denominator, and the equation expressed in integers.

$$x = d \cos (n-d)\,\theta + (n-d) \cos d\,\theta$$
$$y = d \sin (n-d)\,\theta - (n-d) \sin d\,\theta.$$

The practical adjustments of the pen in the Creighton machine for a hypocycloid differ from those given before for an epicycloid (535) only in this, that the pen is set on Y at the value of m in phase $Y = +90°$, instead of $-90°$, the phase of X remaining at 0° as it was before.

Special Cases

542. *First* $m = 1$. Then

$$\begin{aligned} x &= \cos 0° + 0 = +1 \\ y &= \quad 0 \; - 0 = \quad 0 \\ \rho &= \quad 1 \; + 0 = \quad 1. \end{aligned}$$

The hypocycloid of one cusp is therefore the point $+1, 0$. That this is true may be explained mechanically. When the rolling circle is 0.9 as large as the fixed one, the tracing point will, after one revolution,

come back on the latter at 0.1 of the circumference from its starting point. When the rolling circle is 0.99, the difference will be only 0.01, when 0.999 it will be 0.001, and so on, and when 1.000 it will be 0.000.

543. *Second* $m > 2$.
When m is an integer greater than 2, the appearance of the hypocycloid is the ordinary one as given in text books. Thus if $m = 4$, in Fig. 548A,

$$x = \cos 3\,\theta + 3 \cos \theta$$
$$y = \sin 3\,\theta - 3 \sin \theta.$$

By simple trigonometric and algebraic operations the usual equation is obtained

$$x^{2/3} + y^{2/3} = 4^{2/3}$$

The 4, however, is always replaced by an algebraic constant. It is the radius of the fixed circle, m.

When m is an improper fraction greater than 2, such as $5/2$ in Fig. 545A, the figure presents the expected appearance. There are 5 cusps and 2 revolutions of the rolling circle inside the fixed one, the radius of the latter being as usual m. There are also 2 intersections of the curve made by radial lines.

544. *Third* $m = 2$.
The value of $m = 2$ is a critical one. Here, since

$$x = \cos \theta + \cos \theta = 2 \cos \theta$$
$$y = \sin \theta - \sin \theta = 0.$$

the hypocycloid of 2 cusps is a straight line, the diameter of the fixed circle. As the diameter of the rolling circle is now equal to the radius of the fixed one, the circumference of the first will always pass through the center of the second.*

545. *Fourth* $2 > m > 1$.
When $m = 5/3$ as in Fig. 545B, surprises and difficulties are awaiting the inexperienced draughtsman, because the hypocycloid $m = 5/2$ in Fig. A is identical except in size with $m = 5/3$ in Fig. B. In the latter case there are, of course, 5 cusps (= the numerator) but there seem to be only 2 revolutions instead of the 3 which the denominator indicates.

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* In his book "The Science of Musical Sounds," D. C. MILLER gives on page 10 a view of a mechanical contrivance that illustrates this and shows how the tracing point then moves with simple harmonic motion.

The diameter of the rolling circle is now greater than the radius of the fixed one, so that the tracing point must remain on the farther side of the diameter of the latter, while for values of m less than 2 it remains on the near side. The consequence is an apparently reversed direction of the rolling circle, so that while in Fig. A with $m = 5/2$ the tracing point moves from the right cusp towards the *lower* one on

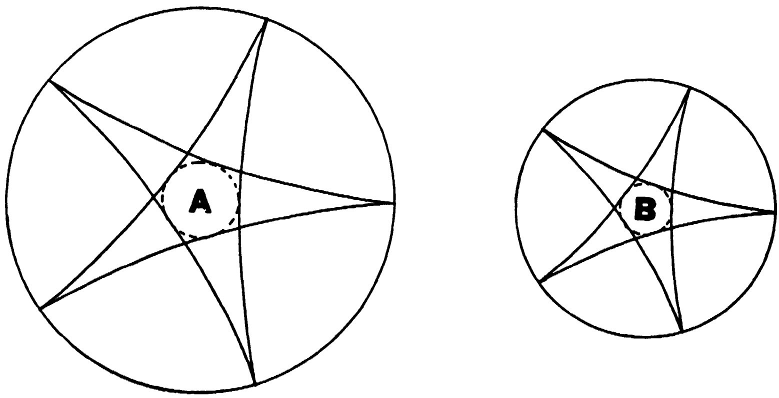


FIG. 545. The Hypocycloids, $m = 5/2$ and $m = 5/3$.

the left side, in Fig. B with $m = 5/3$ it moves towards the *higher* one on the left side. The apparent number of revolutions of the rolling circle seems to be 2 in both cases. It is however only 2 for $m = 5/2$, and really 3 for $m = 5/3$, when the turns of the disk are carefully counted. The denominator of m therefore retains its former meaning here also, as being the number of revolutions of the rolling circle on the fixed one, the numerator, as ever, giving the number of cusps. The number of intersections of the curve by a radial line, is, therefore, no certain index of the number of turns of the disk.

As the diameter of the rolling circle is greater than the radius of the fixed circle, because $2 > m > 1$, $m - 1$ remains positive and a proper fraction. For example if $m = 5/3 = n/d$, then $m - 1 = 2/3 = (n - d)/d$. The rolling circle, because its radius is smaller than that of the fixed one, in one revolution rolls over $3/5 = 1/m = d/n$ of its circumference, that is, over more than half, because $\dfrac{1}{1} > \dfrac{1}{m} > \dfrac{1}{2}$. As the tracing point P cannot cross the diameter of the first cusp, it can run over only $1 - 3/5 = 2/5 = 1 - 1/m = (m - 1) / m = (n - d) / n$ of it to get to the second cusp. In one cusp interval, or one relative

revolution of the rolling circle, this circle runs over $3/5 - 2/5 = 1/5 = d/n - (n-d)/n = (2d-n)/n$ more than the tracing point P, so that for 5 or n cusps, it runs over $n(2d-n)/n = 2d - n$ turns more, or, which is the same thing, the point P loses $2d - n$ turns on the paper. It makes therefore $d - (2d-n) = n - d = 5 - 3 = 2$ turns, that is, the *difference* between the numerator and denominator. This, therefore, is the number of radial intersections, $n - d$, when $2 > m > 1$.

546. The solution of another difficulty about the identity of $m = 5/2$ in Fig. 545A and $m = 5/3$ in Fig. B, has already been hinted at, in that the two figures are not of the same size, although in their generation the radius of the rolling circle was the same as always, one inch. The answer is that the radius of the fixed circle, m, was different and the tracing point travelled on opposite sides of its center. When this tracing point was farthest from the circumference of the latter, it was the distance 2 away, twice the unit radius. Measurement will show that the distance is 2 from the fixed circle to the near part of the curve in Fig. A and to the far part in B, at points where the curve approaches closest to the center of the disk.

Mathematics comes to consolidate the reasoning. The equation of the hypocycloid for $m = 5/2$ in Fig. A is

$$x = \cos\frac{3}{2}\theta + \frac{3}{2}\cos\theta$$

$$y = \sin\frac{3}{2}\theta - \frac{3}{2}\sin\theta$$

Here the radius of the rolling circle, a, is unity, that of the fixed one is $m = 5/2$, the minimum distance from the center of the disk $= m - 2 = 1/2$.

In Fig. B measurement shows that the radius of the fixed circle is m, as it always is, here $5/3$ inch, which is $2/3$ of that in Fig. A, that is, $5/3$ is $2/3$ of $5/2$. The scale in Fig. B for $m = 5/3$ is therefore $2/3$ as great as before. Therefore the radius of the rolling circle is no longer unity, but $2/3$, and this value must be put into the equation (526).

In addition, the rolling circle in Fig. B makes 3 ($= d$) revolutions instead of the 2 it made in Fig. A. It therefore turns $2/3$ as fast, and this factor must go into θ. This is $2/3$ and not $3/2$, as might at first be imagined, because one out of the 3 turns necessary will not bring the tracing point as far as one out of 2. Hence the rotation is slower, and must be $2/3$. This factor $2/3$ is then to multiply the coefficients of both the terms and the angles.

$$x = \frac{2}{3} \cos \frac{2}{3} \cdot \frac{3}{2} \theta + \frac{2}{3} \cdot \frac{3}{2} \cos \frac{2}{3} \theta$$

$$y = \frac{2}{3} \sin \frac{2}{3} \cdot \frac{3}{2} \theta - \frac{2}{3} \cdot \frac{3}{2} \sin \frac{2}{3} \theta$$

$$\text{or } x = \frac{2}{3} \cos \theta + \cos \frac{2}{3} \theta$$

$$y = \frac{2}{3} \sin \theta - \sin \frac{2}{3} \theta$$

which is the equation of a hypocycloid for $m = 5/3$, and in which the reversal of the signs of both the y terms indicates a reversal in the direction of the revolution of both circles. But this is non-essential. Therefore the hypocycloid $m = 5/2$ and $m = 5/3$ are equal when the radii of their rolling circles are taken as the unit of measurement. Their absolute sizes are proportional to their m's, as must always be the case. When however the radius of the fixed circle m is taken as the unit of measurement, then the radius of the rolling circle will vary inversely as m. In the scale here adopted, m was equal to $5/2$ and $5/3$, while a was unity. In the new scale with the radius of the fixed circle $= ma$, (526), as unity, a would be equal to $2/5$ and $3/5$. The general equation of an epicycloid and hypocycloid would be the same as before, except that $1/m$ would replace a in (238) or (526), so that it becomes for an epicycloid

$$x = -\frac{1}{m} \cos (m + 1) \theta + (1 + \frac{1}{m}) \cos \theta$$

$$y = \frac{1}{m} \sin (m + 1) \theta - (1 + \frac{1}{m}) \sin \theta$$

and for a hypocycloid

$$x = \frac{1}{m} \cos (m - 1) \theta + (1 - \frac{1}{m}) \cos \theta$$

$$y = \frac{1}{m} \sin (m - 1) \theta - (1 - \frac{1}{m}) \sin \theta.$$

This new scale, with the radius of the fixed circle as the unit of measurement has great mechanical inconveniences on the CREIGHTON machine. It may not have them on other machines.

On both scales, m is the number of cusps per revolution of the rolling circle, the coefficient (or the amplitude) of the first terms in x and y is the radius of the rolling circle, and that of the second terms

the radius of the center-circle, that is, the circle in which the center of the rolling circle moves (524). The radius of the rolling circle is the unit of measurement in the first case, and that of the fixed circle in the second, these radii being to each other as $1:m$ and as $^1/m:1$ respectively in the two systems. This subject will be resumed later (561).

547. Further study shows that the above conclusion in regard to the two Figs. 545 is true only when the sum of the two denominators 2 and 3 in $m = 5/2$ and $m' = 5/3$ is equal to the numerator 5, so that in general, the hypocycloids $m = n/d$ and $m' = n/(n-d)$ are similar. Hence the curve $m = 5 = 5/1$ must be similar to $m' = 5/(5-1) = 5/4$ in Fig. 548C, but four times as large. The figure for $m = 5$ was not drawn because it was too large for the disk (536). In the same way $m = 13/11$ as in Fig. 548B is similar to $m' = 13/2$. This last came as a great surprise to the writer, because he had expected 11 radial intersections and got only 2. The disk however rotated 11 times, so that

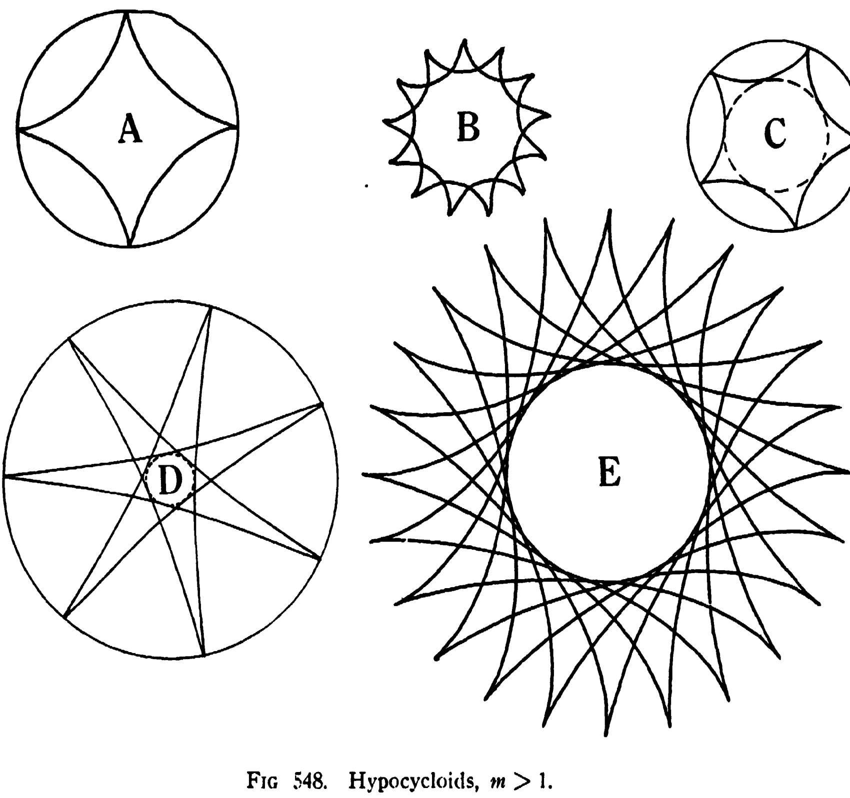


FIG 548. Hypocycloids, $m > 1$.

the pen moved very slowly on the paper. The number of radial intersections is therefore the smaller number of d and $n - d$.

548. When the value of m is near 2, as in $7/3$ in Fig. 548D, the cuspidal arms are sharp and like prongs. When it is far away from 2, the figure looks like a sprocket wheel, as in Fig. B, where $m = 13/11$ or $13/2$. The best values of m to take, when fine figures are desired, are those with large numbers in the numerator and denominator, such as $m = 24/7$ in E, taking care, however, that d is not near $n/2$.

551. *Fifth* $m < 1$.

If hypocycloids with values of m greater than 1 are quite a surprise to one who investigates them the first time, those with $m < 1$ are much more so, since these hypocycloids seem to be and are truly epicycloids. Indeed, they look exactly like the latter and not at all like the former, because their cusps face inward instead of outward. When $m < 1$, the fixed circle m is smaller than the rolling one, or better conversely, the rolling circle is larger than the fixed one, so that the tracing point cannot rotate on the inside of it, but must trace an epicycloid *about* it.

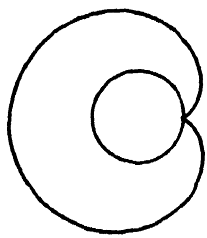


FIG. 551. The Hypocycloid $m = 1/2$.

Beginning with the value $m = 1/2$ in Fig. 551, the curve will be found to be a *cardioid*. Here $m - 1 = -1/2$, hence

$$x = \cos 1/2\, \theta - 1/2 \cos \theta$$
$$y = -\sin 1/2\, \theta + 1/2 \sin \theta.$$

The proof is interesting, but too long to insert here. By shifting the origin to $+1/2$, 0, the polar equation is

$$\rho = 1 - \cos 1/2\, \theta.$$

This hypocycloid $m = 1/2$ is a dividing line between values of m greater and smaller, which remain however less than unity, just as $m = 2$ separated hypocycloids of greater and less values. When $m = 1/2$

the circle in which the center of the rolling one moves, coincides with the fixed one, and when $m > 1/2$ it is less or within it, and when $m < 1/2$ it is greater or outside of it. Another and more important difference will be seen later.

552. Let $m = 2/5 < 1/2$ as in Fig. 552A. The numerator 2 shows the number of cusps, as it always does, the denominator 5 the number of

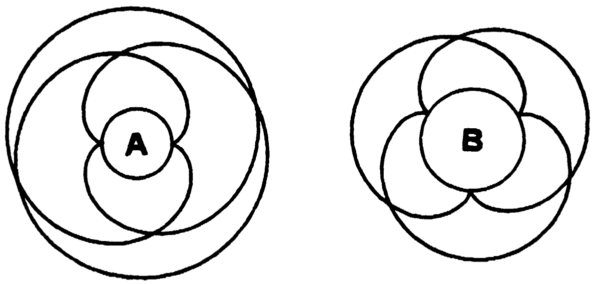


FIG. 552. The Hypocycloids $m = 2/5$ and $m = 3/5$.

turns of the disk or the revolutions of the rolling circle about the fixed one. The equation, since $m - 1 = -3/5$, is

$$\begin{aligned}x &= \cos 3/5\, \theta - 3/5 \cos \theta \\ y &= -\sin 3/5\, \theta + 3/5 \sin \theta \\ \varrho^2 &= 1 + (-3/5)^2 - 2 \cdot 3/5 \cos 2/5\, \theta.\end{aligned}$$

As the figure looks exactly like an epicycloid, one is tempted to try the same value of $m = 2/5$ in its equation. This however gives a totally different answer on account of the $m + 1$ replacing the $m - 1$. Then, when m is taken equal to $-2/5$

$$\begin{aligned}x &= -\cos 3/5\, \theta + 3/5 \cos \theta \\ y &= \sin 3/5\, \theta - 3/5 \sin \theta \\ \varrho^2 &= 1 + (+3/5)^2 - 2 \cdot 3/5 \cos 2/5\, \theta.\end{aligned}$$

This equation is identical with that given above, except that all the signs are changed. But as this affects only the direction of measurement or the direction of rotation, and leaves the polar equation absolutely identical, one feels elated over a brilliant discovery, and proclaims that hypocycloids and epicycloids with the same values of m less than unity, but of opposite signs, are identical.

As the excitement calms down, the thought comes that reversing the sign of m has no meaning, because m is an absolute constant, a ratio, and that substituting $-m$ in the equation of an epicycloid changes

$m+1$ into $-m+1=-(m-1)$, and thereby reverses all the signs in the equation of an epicycloid and makes a hypocycloid out of it. The reverse is also true.

In calling upon the machine to decide the issue, the hypocycloid $m=2/5$ in Fig. 552A and the epicycloid $m=2/5$ in Fig. 534B seem at a first glance to be identical except in size. They have the same m, and therefore the same fixed circle. The maximum radius vector of the first, however, is $m-2=-8/5$, and that of the second $m+2=12/5$, a difference of $2m$, and worst of all, the convolutions of the first are only 3, whereas those of the second are 5. Hypocycloids and epicycloids with the same value of m less than unity, but of opposite sign, cannot therefore be identical. The hypo $3/5$ in Fig. 552B and the epi $3/5$ in Fig. 534A prove the same thing.

553. Stopping here would allow a most interesting and real discovery to escape. Study shows that the epicycloid $m=3/5$ in Fig. 534A has 5 radial intersections, a number equal to d, the denominator of m, and thereby conforms to the general law for epicycloids. But the hypocycloid $m=3/5$ in Fig. 552B has only 2 radial intersections $=d-$n. As in this case $m<1$, the rolling circle is larger than the fixed one, so that its center and the point of contact must always be on opposite sides of the center of the fixed circle. When the rolling circle in Fig. 552B rolls from the first cusp to the second, it rolls over $5/3=1/m=d/n$ of the circumference of the fixed circle, or $5/3-1=2/3=(d/n)-1$ $=(d-n)/n$ more than one complete turn. Because $m<1$, and the center of the rolling circle is 180° away from the point of contact, both circles, the rolling one and the one its center travels in, revolve in the same rotary direction, the same as in the case of all epicycloids (527), and the reverse of that in hypocycloids when $m>1$.

The consequence is that the tracing point moves in the same direction, but loses one revolution thereby, so that it moves on the paper only over $5/3-1=2/3=(d-n)/n$ of a complete turn. As there are 3 or n cusps, it must move over 3 times $2/3=2=n\,(d-n)/n=d-n$ turns and make therefore $d-n$ radial intersections, which is the difference between the denominator and the numerator.

This may be substantiated mathematically. From the first cusp to the second the rolling circle rolls over $5/3$ of $360° = \frac{1}{m}\,360° = \frac{d}{n}\,360°$. As $m-1=-2/5=(d-n)\,/\,d$, one whole turn of the rolling circle is $\frac{d-n}{d}\cdot\frac{d}{n}\,360° = \frac{d-n}{n}\,360° = (\frac{d}{n}-1)\,360°$, which, as $n\,/\,d=$

$m < 1$, and hence $d / n > 1$, means more or less than one revolution according as d / n is greater or less than 2 or $n / d = m$ less or greater than $\frac{1}{2}$. Thus in Fig. 552B with $m = \frac{3}{5}$, the cusp interval or one revolution is less than 360°, while in Fig. 552A with $m = \frac{2}{5}$ is is more, and in Fig. 551 with $m = \frac{1}{2}$ it is exactly one.

It is therefore universally true that every epicycloid whatever, whether its m is greater or less than unity, is identical with a hypocycloid whose m is less than 1. That this last condition must be verified is shown by the fact that the m of the hypocycloid is equal to $n/(n+d)$ of the epicycloid, and this fraction is evidently less than unity. The scale however is not the same.

Fig. 555A was actually drawn as a hypo $m = \frac{4}{5} = 4/(4+1)$. It is $\frac{1}{5} = (d-n)/n$ of the scale of the epi $m = \frac{4}{1}$ in Fig. 533A. Fig. 555B also was drawn as a hypo $m = \frac{3}{4}$. It is $\frac{1}{4}$ of the scale of the epi $m = 3$.

554. As hypocycloids that have $m > 1$ may be recognized as such at sight, because their cusps point outward, as in Figs. 545 and 548, the difficulty of identification can arise only from the apparent similarity between hypocycloids with $m < 1$ and all epicycloids, both classes of curves having their cusps pointing inward.

Let Fig. 554 be taken as a test case, and let nothing be known about it except what the figure itself will disclose. The question to

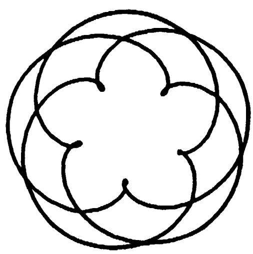


Fig. 554. Epicycloid or Hypocycloid?

be answered is therefore whether this is a hypocycloid with its $m < 1$ or an epicycloid, or both. If it is an epicycloid, as the number of cusps is $5 = n$, and the number of radial intersections is $3 = d$, m must be equal to $n/d = \frac{5}{3}$. Now, as the maximum radius vector must be $m + 2$, the minimum and maximum radii must be in the ratio of $m : m + 2$, or $\frac{5}{3} : \frac{5}{3} + 2$ or $5 : 11$. Measurement on an arbitrary scale gave $18 : 40$, which is close enough to confirm this.

Testing it as a hypocycloid, $n = 5$ as before, but $d - n = 3$, and therefore $d = n + 3$ and $m = n/d = \frac{5}{8}$. The minimum radius vector

must be $m = 5/8$ and the maximum $2 - m = 2 - 5/8 = 11/8$, a ratio of 5 : 11, which is the same as before. This curve is therefore both an epicycloid with $m = 5/3$ and a hypocycloid with $m' = 5/8$.

It is plain that the number of cusps $= n = 5$ must be the same in both. If the denominator of the epi m is called d, $m = n/d$, that of the hypo is $d' = n + d$, so that $m' = n/(n + d)$. As the radial intersections of the epi are $d = 3$, those of the hypo as shown before must be $(n + d) - n = (5 + 3) - 5 = 3 = d$, so that this fact confirms the double nature of the curve in question as being both an epi and a hypo.

555. This may be proved mathematically and thus put on a sound basis. Taking the equation of an epicycloid (531)

$$x = -d \cos (n + d)\, \theta + (n + d) \cos d\, \theta$$
$$y = d \sin (n + d)\, \theta - (n + d) \sin d\, \theta$$

and replacing its d by $d - n$ or $-(n - d)$, and $n + d$ by d, (because $n + (d - n) = d$), it is found that

$$x = (n - d) \cos d\, \theta + d \cos (n - d)\, \theta$$
$$y = -(n - d) \sin d\, \theta + d \sin (n - d)\, \theta$$

which is the equation of a real hypocycloid (541). The converse proposition may be proved with equal facility.

Fig. 555A is an epi $m = 4$ and a hypo $m = 4/5$. 555B is an epi $m = 3$, and a hypo $m = 3/4$. 533B is an epi $m = 6/5$ and a hypo $m = 6/11$.

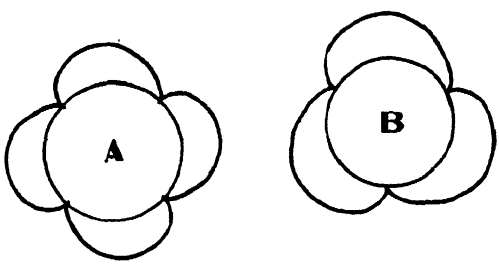


FIG. 555. The Hypocycloids $m = 4/5$ and $m = 3/4$ are reduced Epicycloids $m = 4$ and $m = 3$.

533C is an epi $26/25$ and a hypo $26/51$. 534E is an epi $44/45$ and a hypo $44/89$, and so on. The identity of all epicycloids with hypocycloids $m < 1$, except as to scale, may be used in machinery in tracing the curve either as one or as the other according as convenience or necessity may require.

561. It was some time after all the above had been written that the writer succeeded in obtaining the book "A Treatise on Cycloids" by R. A. PROCTOR, London, Longmans, Green and Co., 1878. This work is so rare that even the publishers did not have a copy. The one obtained was due to the courtesy of the librarian of the University of Nebraska. PROCTOR takes exception to the unequal division generally made between epicycloids and hypocycloids, and to the statement that a large class of curves should have the double name of both epicycloids and hypocycloids. Although he does not use the expression, he restricts the names of epicycloids and hypocycloids to those circular cycloids that have their cusps facing respectively inward and outward. Instead of the words "external and internal contact" (531, 541), PROCTOR gives the definition in this way: "An epicycloid (or hypocycloid) is a curve traced by a point on the circumference of a circle which rolls on a fixed circle so as to touch its outside (or inside)." According to this, no curve can be both an epicycloid and a hypocycloid, that is, hypocycloids with $m < 1$ should be classed as epicycloids. And this seems more reasonable.

562. The two figures here given, 562 and 563, together with their explanations, have been taken from PROCTOR with considerable modifications. In these he proves what has been detailed above in regard

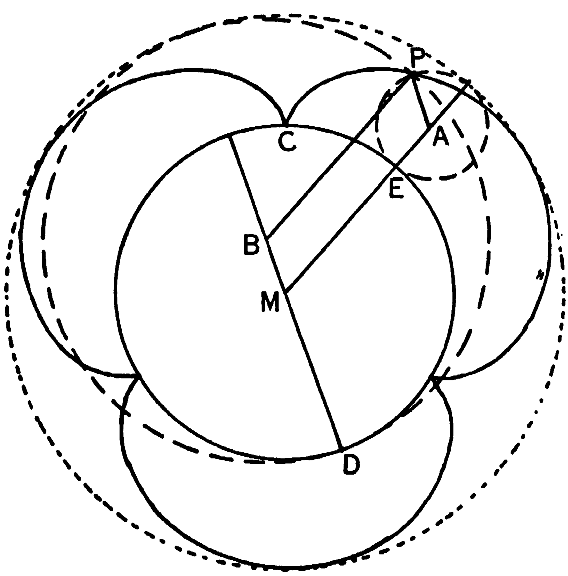


FIG. 562. Proctor's Proof.

to hypocycloids in 545-548 when $1 < m < 2$ and in 551-555 when $m < 1$.

EPICYCLOID. According to the usual as well as to PROCTOR's definition the dashed circle A in Fig. 562 rolling on the full-line circle M generates an epicycloid, CP. . . . The same curve CP . . . is generated by the circle B which, as PROCTOR says, touches the outside of the fixed circle M, or according to the usual parlance, preserves internal contact with M. This fact is to be proved.

When the A circle has rolled from C and touches the M circle at E, its tracing point is at P. For the same point P the B circle has also rolled from C in the same or clockwise direction and touches the M circle at D. Let the radius of the A circle be $a = 1$, of the B circle $b = 4$, and of the fixed circle M $m = 3$. Let the arc CE on the M circle $= 40°$, then its equal PE on the A circle $= 3 \times 40° = 120°$. The arc CD on the M circle is then $4 \times 40° = 160°$, and PD on the B circle $= 3 \times 40° = 120°$, as measured on their own circles. As the arc $ED = CD - CE = 160° - 40° = 120°$, the angles $AMD = PBD$, and the lines PB and AM are parallel. They are also equal, because $AM = 1 + 3$ and $PB = 4$. Therefore PA is equal and parallel to BM. For another reason $AP = 1 = BD - MD = 4 - 3$.

To generalize this statement and to prove that the A and B circles generate the same cycloid, PROCTOR begins by supposing that the B circle alone draws it. He draws MD through the point of contact D, and then the line MA parallel and equal to BP. $MBPA$ is then a parallelogram and $AP = MB$, or $a = b - m$. P is therefore on the A circle. The arcs PD and CD are given equal, but as they are in different circles, the angles they subtend are inversely as the radii. Hence angles $PBD : CMD :: m : b$. As $PBD = EMD = PAE$

$$\frac{PAE}{CMD - EMD} = \frac{m}{b - m} = \frac{PAE}{CME} = \frac{m}{a}.$$

Therefore the arcs $PE = CE$ and the A circle generates the same cycloid.

563. *HYPOCYCLOID.* In regard to the hypocycloid in Fig. 563, the proof may be given almost in the very same words that have been used for the epicycloid. But because the numerical quantities are different, it will be well to repeat it.

The hypocycloid in this Fig. 563, the five-pointed star, is generated by both the A and B circles (545-547). When the A circle has

rolled from C in an anticlockwise direction and touches the M circle at E, its tracing point is at P. For the same point P the B circle has also rolled from C, but in the opposite or clockwise direction, and touches the M circle at D. Let the radius of the A circle be $a = 2$, of the B circle $b = 3$, and of the fixed M circle $m = 5$. Let the arc CE

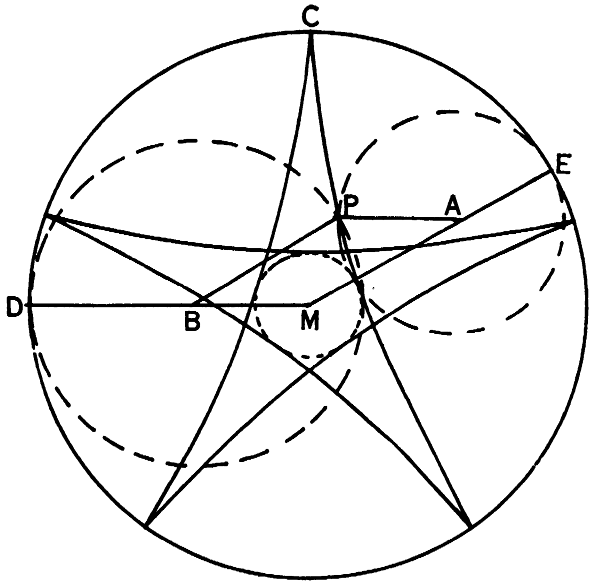


FIG. 563. Proctor's Proof.

on the M circle be 60°, then its equal PE on the A circle $= 5/2 \times 60° =$ 150°, the arc CD on the M circle is then $3/2 \times 60° = 90°$, and PD on the B circle is $5/3 \times 90° = 150°$, as measured on their own circles. As the arc $ED = DC + CE = 90° + 60° = 150°$, the angles $AMD = PBD$, and the lines PB and AM are parallel. These are also equal because $AM = 5 - 2$ and $PB = b = 3$. Therefore PA is equal and parallel to BM. For another reason AP $= 2$ and $MD - BD = 5 - 3$.

To generalize this statement, after supposing that the B circle alone traces the hypocycloid, MD is drawn through the point of contact D, and then the line MA parallel and equal to BP. $MBPA$ is then a parallelogram and $AP = MB$, $a = m - b$. P is therefore on the A circle. The arcs PD and CD are given equal, but as they are on different circles, the angles they subtend are inversely as the radii. Hence the angles $PBD : CMD :: m : b$. As $PBD = EMD = PAE$,

$$\frac{PAE}{EMD - CMD} = \frac{m}{m-b} = \frac{PAE}{CME} = \frac{m}{a}.$$

Therefore the arc $PE = CE$, and the A circle generates the same hypocycloid.

The radius of the fixed circle m in the case of the epicycloid is equal to the difference $b - a$ of the radii of the rolling circles, and in the case of the hypocycloid it is equal to their sum $b + a$.

Summary of Chapter V

(511) A *cycloid* is defined to be the curve traced by a point on a circle while the center of this circle moves in a second one. In the same way there may be a third, a fourth, and more circles similarly related.

(513) A *common cycloid* with its curtate and prolate forms.

Circular cycloids (521-523) have the equations (524) given before in (237) and below. Reciprocity in (527).

Epicycloids (531) have the equation

$$x = -\cos(m+1)\theta + (m+1)\cos\theta$$
$$y = \sin(m+1)\theta - (m+1)\sin\theta$$
$$\rho^2 = 1 + (m+1)^2 - 2(m+1)\cos m\theta$$

in which the radius of the rolling circle is unity, and m that of the fixed circle on which it rolls and also the number of cusps per revolution. When $m = n/d$,

$$x = -d\cos(n+d)\theta + (n+d)\cos d\theta$$
$$y = d\sin(n+d)\theta - (n+d)\sin d\theta.$$

Special Cases. When $m = 1$ (532) the curve is a cardioid. When $m > 1$ (533), and when $m < 1$ (534), illustrations are given and methods of drawing (535, 536).

Hypocycloids (541) have the equation

$$x = \cos(m-1)\theta + (m-1)\cos\theta$$
$$y = \sin(m-1)\theta - (m-1)\sin\theta$$
$$\rho^2 = 1 + (m-1)^2 + 2(m-1)\cos m\theta$$

in which the radius of the rolling circle is unity, and m as given before for epicycloids. When $m = n/d$,

$$x = d\cos(n-d)\theta + (n-d)\cos d\theta$$
$$y = d\sin(n-d)\theta - (n-d)\sin d\theta.$$

Special Cases.

When $m = 1$ (542), the curve is a point.
When $m > 2$ (543).
When $m = 2$ (544), the curve is a straight line.
When $2 > m > 1$ (545-548) anomalies and difficulties appear.
When $m < 1$ (551-555), the hypocycloids become epicycloids.
PROCTOR'S *definition* (561-563) and proofs.

CHAPTER VI

BEAUTY. ARTIFICES, AND SURPRISES

I. Beauty

611. The universal fondness for harmonic curves is evidently due to their beauty. There is also first the curiosity to know what the figure will turn out to be. Then the fascination to see it grow, watch it develop, and to be kept in suspense in regard to its final shape often to the very end. And lastly, there is frequently the captivation, amounting at times almost to excitement, to follow the pen on its last round as it glides majestically right through the middle of the lane left vacant for it, accurately back to the starting point.

612. Tastes, of course, will differ, and there will be much diversity in the selection of the finest curves that a machine has produced. But they are all beautiful when well drawn, there is question only of the degree. On account of this variety in taste it would be futile to make the curves amenable to law. Still, there are certain rules and directions for the formation of fine figures, as well as of refined taste, certain hints and artifices which a draughtsman may employ at once instead of laboriously learning them from experience. A few of these are here subjoined.

613. One of these rules is that, within certain limits, of course, a curve gains in beauty in proportion to the number of cycles that the pen has in one axis or in both, and the turns that the disk makes. Compare Figs. 713, 717, 743C, 951, 955, 958, and 625 A and B, 724, 929 D and E, etc. As a general thing the disk should make relatively few turns, not only as in Figs. 548 A, 625 E, G, I, 924 B, C, in all of which, however, except the first, this draw-back is offset by the many cycles in the pen; nor too many as in Figs. 534 C, D, E; but just so many that there may be an agreeable number of intersections as in Figs. 534C, 548 B, E, 625 A, B. Other suggestions will be given later (625, 626).

614. The statement that all harmonic curves are beautiful may be open to objection. Sine curves would most probably not be called so. It is their utility that is appreciated by mathematicians and scientists. Many other curves also would be rejected on the score of beauty. Every

draughtsman knows this. So much so, that while his experience may give him a kind of intuition as to the probable beauty of the curve he intends to draw, this foreknowledge is not always certain. The curve may turn out to be one that, he fears, would not be a credit to his machine.

II. Artifices

621. *Circles* may be drawn by a disk and by equal X and Y components differing 90° in phase. The way is thus opened to all kinds of circular cycloids. Then by merely altering an initial phase, the circle may be changed into an ellipse or a straight line (941).

622. *Variable Circles and Ellipses.* With two pairs of equal X and Y components differing in phase and having slightly unequal periods, such as 44 and 45, a circle or an ellipse may be made to grow from a point to a maximum size and diminish again to the point (934).

623. *The Direction of the Rotation* of two circles is the same when one component, such as X, in each is 90° ahead (or slow) of the other, but when this component is ahead in one pair and behind in the other, the circles rotate in opposite directions (223).

624. *A Variable Line.* A line may be made to vary in length from zero to a certain maximum by having two components in Y only, equal in amplitude and slightly unequal in period (925).

625. *Joint Curves.* Two or more curves may be placed together on the same paper. When they do not overlap, it is an easy task to mount them. The Frontispiece and Fig. 625 A and C are examples of this class. But when the curves overlap or intertwine, it is a matter of prudence to draw them first separately, to place one over the other, hold the two to a window pane, and study the joint effect. If this is satisfactory, the two curves may be drawn on the same paper. One way of doing this and making many duplicates of the composite is to draw any desirable number of the first or A curve by leaving the drawn figures on the disk by carefully placing new sheets of paper over them without disturbing those underneath. On a fresh sheet of paper the second or B curve is now drawn. It is taken off the disk together with the uppermost A figure, and the joint effect examined. When the judgment is favorable, the B curve is drawn successively on all the A figures.

Duplication or even multiplication of the same or different figures on the same paper was the method frequently employed by Pere DECHEVRENS (451), and is probably the chief reason of the great beauty of his figures. A change of initial phase in the same or in the second curve, or in the period of the disk, or the position of the pen, are additional suggestions that he gives. In one case he drew 48 separate figures differing successively by the same amount in initial phase. Placing these in order and spacing them evenly, any pair could be used stereoscopically throughout the whole series from the first back again to it. As an invalid for many years he had the necessary leisure, and more than that, he had the skill. He has sent the writer many hundreds of his figures, which, it is to be hoped, will some day appear in print.

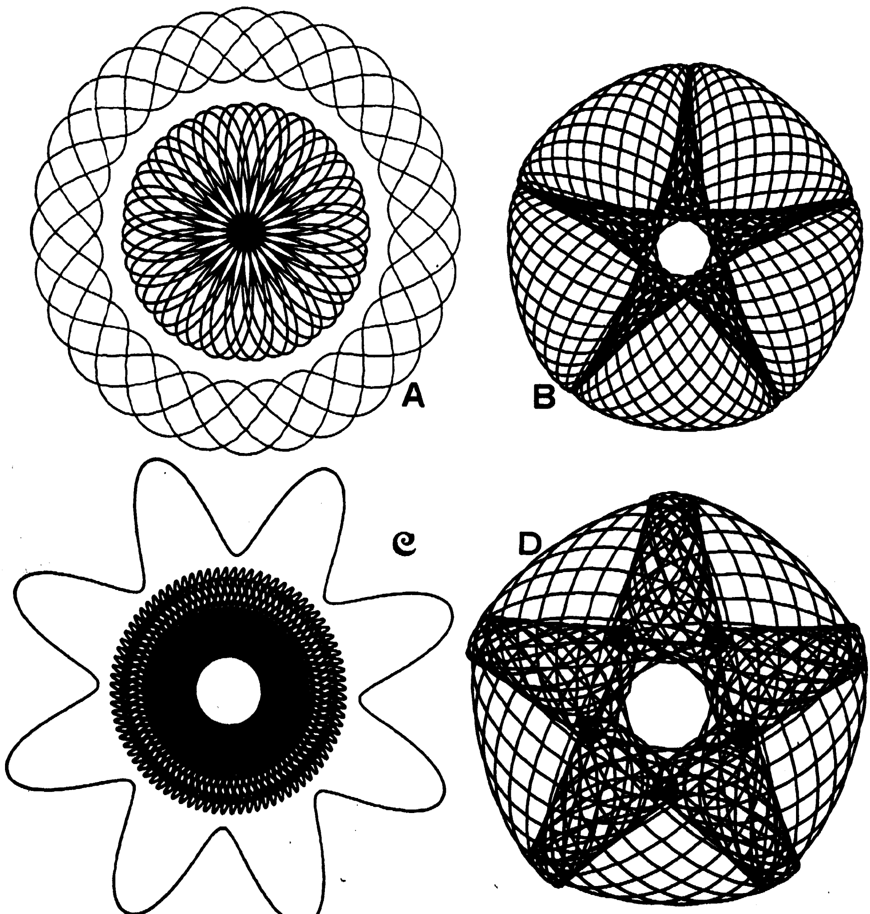


FIG. 625. Devices for Obtaining Beauty.

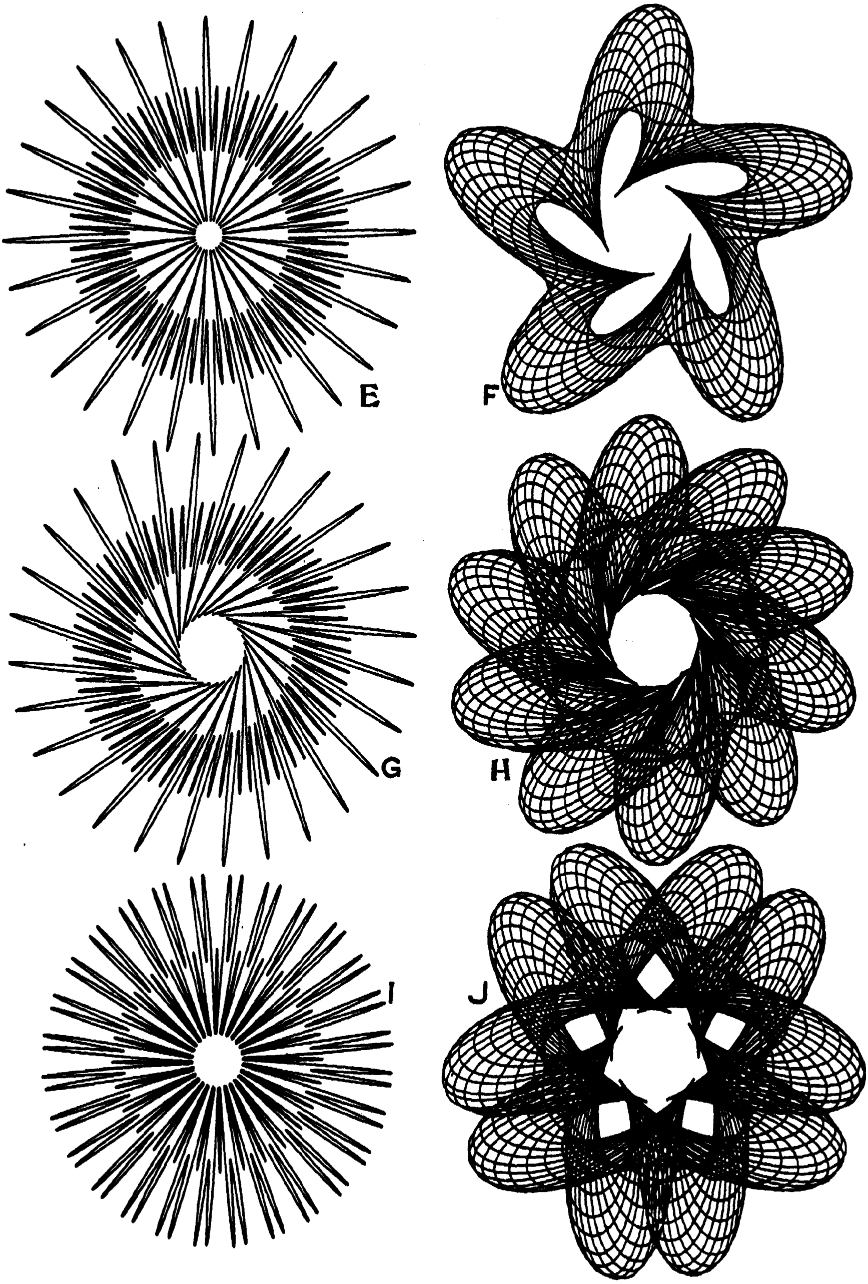


FIG. 625. Devices for Obtaining Beauty.

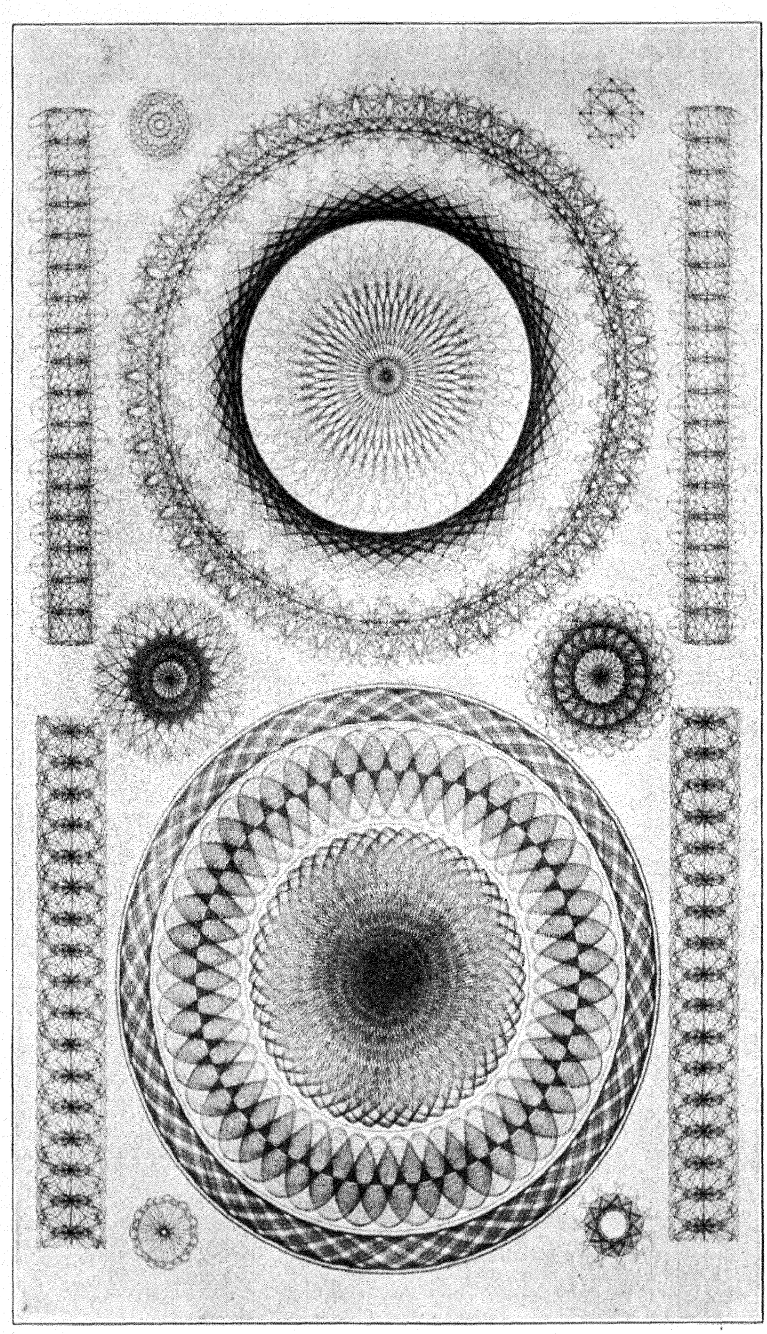


FIG. 625K. A few Dechevrens Figures.

Fig. 625K shows a few of them. They are practically all polar, very few being rectangular. His machine (451) lent itself well to a rapid change of amplitudes without disturbing the zero position of the pen, which the CREIGHTON machine does not. One application of this may be imagined in Fig. 213, in which two simple sine curves are shown differing only in amplitude. This amplitude may be varied in successive steps. The compound figure may also be drawn on a disk, and repeated by an initial rotation of the disk through a fraction of a wavelength. This last artifice gave rise to beautiful shadings. The same was applied to ribbon curves by shifting the paper in equal steps after drawing the same curve over and over again, so that the separate figures partially overlapped. Except in the case of the common cycloid and in a very few simple sine curves, this is the only way he moved his paper ribbon, that is, intermittently, not with uniform speed. A few times he used this method in polar curves. A circle was drawn in pencil (to be erased later) on a sheet of paper, and equidistant points on its circumference marked. Placing these in succession at the center of the disk, he drew the same polar curve about them so that adjacent figures overlapped.

In Fig. 625 some of these suggestions are illustrated. All of the figures are Sine-Polars, with components in Y only over a rotating disk. In the two double ones that do not overlap, A and C, and that have already been referred to, Y had only one component, the ratio of its cycles to those of the disk being 24:7 in the inner curve and 28:5 in the outer one in A, and 84:5 in the inner one and 8:1 in the outer one in C. In E, G, I, Y had two components of periods 84 and 28 in one rotation of the disk. In G the pen moved on a non-radial line, and in I the initial phases were changed. In B, D, F, H, J, the pen had two components of periods 16 and 15. The difference between B and D is merely one of the initial position of the pen. F is a skew pentafolium (See Appendix, Cuspidal Envelope Rosettes). In H the figure F was drawn a second time after the disk had been given an initial turn of 36°. In J the same figure F was also drawn a second time, but in a reversed form. As a matter of interest it may be stated that two pens were used simultaneously in J, one for each figure.

626. *Colored Inks.* Additional beauty may be given the drawings by using inks of different color in its parts or in superposed figures. In his book "Experimental Science," HOPKINS says in vol. II, p. 136, that instead of smoking the glass disk when the figure is to be projected on a screen, one side may be covered with a thin collodion solution of a red aniline color, and the other with a blue one. The stylus will then

draw a red or a blue design in a purple field. When the glass disk is reversed and a different figure traced, the resultant will be a very intricate one in red, white, and blue on a purple background. The second figure, when sharply focussed, will appear to stand out several inches from the screen, and seem to float in the air.

627. *Incomplete Figures.* The reader will probably be much interested in incomplete figures, because all those that appear in print are complete

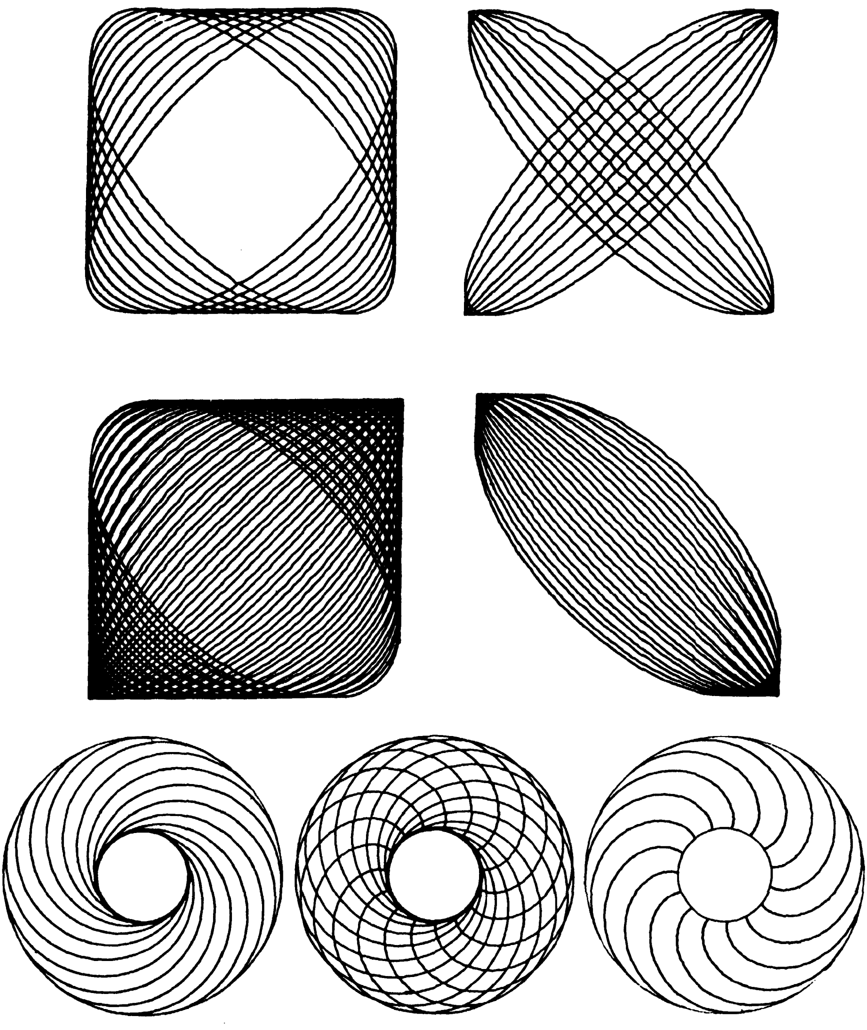


FIG. 627. Incomplete Curves.

ones. These incomplete figures, as said before (611), often give no idea of what the finished product will be. Thus in Fig. 627 the four upper figures are parts of Fig. 958. In the lower three the middle one is complete (15 cycles of Y to 16 of the disk), while the right and left ones are its complementary halves.

III. Surprises

While beauty is the chief attraction in harmonic curves, the surprises they occasion are probably their second alluring factor. Thus the whole subject of cycloids (Chapter V) was a revelation to the writer, together with many other items. But a few surprises of another sort are set down here.

641. The pen is made to draw simultaneously two equal circles in the same direction and with equal speeds. What is the result?

The answer is so obvious to almost all that they will reply at once that the pen draws the unit circle with twice the speed. This is, however, not correct, because the equation of the pen's motion is (223)

$$x = \cos\theta + \cos\theta = 2\cos\theta$$
$$y = \sin\theta + \sin\theta = 2\sin\theta$$

which shows that the amplitude is doubled, and not the speed, that is, the pen describes a circle with twice the radius but with the same angular speed. The "obvious" answer given at first is correct mechanically, because it supposes the two circles to have the same center (523). Harmonically, however, the center of the second circle is on the circumference of the first, so that it is the co-ordinates that must be added, and not the speeds. The starting phases must also be the same, so that the circles will always remain in phase. This suggests a way of doubling and, in general, of increasing an amplitude, should this be found convenient or necessary, as when it is beyond the ordinary reach of the machine.

642. The pen is made to draw two circles with the same radius, in the same direction, but with unequal speeds. What is the curve?

From what has been said in the foregoing case, it is seen that the co-ordinates are to be added, so that the equation is

$$x = \cos\theta + \cos m\theta$$
$$y = \sin\theta + \sin m\theta.$$

The resultant is therefore not a circle. When m is very little less than unity, then as θ is very small, the pen begins to draw a circle with

radius 2. As θ grows, the sum of $\cos\theta + \cos m\theta$ or x is less than $2\cos\theta$, and $\sin\theta + \sin m\theta$ or y is less than $2\sin\theta$, so that the pen draws a circle with radius less than 2. This radius is constantly diminishing, although not uniformly. It becomes zero, together with x and y, when the two circles differ 180° in phase, so that $\cos\theta = -\cos m\theta$ and $\sin\theta = -\sin m\theta$. The pen is then at rest. It is very interesting at this time to look at the machine with all its gear wheels rotating at the usual speed and the flywheel spinning round as fast as ever, but with the pen perfectly still and motionless, just as if it had somehow or other been disconnected from the machinery. A look at the sections on the same side will then show one going up as fast as the other is going down, one pulling the cord as fast as the other is paying it out, so that the algebraic sum of their movements is zero. This may last practically for almost half a minute or more, theoretically of course only for a moment. Then the pen slowly resumes its motion, describing continually larger circles always in the same direction as before, until it returns to the starting point and closes the curve. Fig. 934 is the result.

643. The pen is forced to draw two equal circles with the same speed, but in opposite directions. What will happen?

The answer to this question was at first even more obvious than to the one in 641, so very obvious in fact that the writer did not think it worth while to try it on the machine, because he was convinced that there would be a deadlock and the pen would remain stationary. At last, after harboring this conviction for a month, he tried the case following this one with the speeds slightly unequal. The result then indicated the solution of the present one.

There is the same misconception here as in Fig. 641, in supposing that the circles have the same center. This is not the case, because harmonically the center of one circle is on the circumference of the other, so that it is the co-ordinates that are to be added algebraically, and not the speeds. The equation is (223)

$$
\begin{aligned}
x &= \cos\theta + \cos\theta = 2\cos\theta \\
y &= \sin\theta - - \sin\theta = 0, \\
\text{or}\quad x &= \cos\theta - \cos\theta = 0 \\
y &= \sin\theta + \sin\theta = 2\sin\theta.
\end{aligned}
$$

In both cases the curve is a straight line on one of the axes, equal in length to 4. It is a hypocycloid of 2 cusps (544).

When the two circles are given the same center (523), the curve is the expected point. This cannot be done in the rectangular way, but the point may be drawn by taking four components equal in amplitude and in period, two in Y and two in X, those in X starting with $+90°$ and $-90°$ and those in Y with $0°$ and $180°$ respectively. Then the pen will actually stand still all the time because the sum of the co-ordinates is always zero. In the rectangular-polar method however the two circles may be really made concentric when two components, one on each axis, are taken equal in amplitude and period, but right-angled in phase, and the pen set down at the distance of the radius from the center in the proper position, such as in $+Y$ when $y = 90°$ and $x = 0°$. The disk is then turned with the same period in the direction in which the pen is moving. The pen will then be stationary in relation to the paper, and will seem to be carried around by the disk just as if the rectangular mechanism was out of order. The curve in question, that is, the point, is then a hypocycloid of one cusp (542).

644. The pen draws two equal circles in opposite directions with slightly unequal speeds. What is the result?

Almost everybody, to whom this problem is proposed for the first time, will answer promptly that the pen will draw a unit circle with the difference of the speeds. This is true mechanically when the two circles have the same center. But harmonically when the center of one circle is on the circumference of the other, and when consequently the co-ordinates are to be added, this cannot be the solution.

The equation is

$$x = \cos\theta + \cos m\,\theta$$
$$y = \sin\theta - \sin m\,\theta$$

in which m differs from unity by a small amount.

When a machine is called upon for the solution, the pen will appear to draw a straight line equal in length to 4, and this line will slowly turn about its midpoint. See 924 for further details. Chapter IX contains many other surprises.

Summary of Chapter VI

Beauty. The universal fondness for harmonic curves is due to their beauty (611-614).

Artifices. Methods are given for drawing circles (621), variable circles and ellipses (622), of determining the direction of rotation (623), for drawing a line of variable length (624), for placing one curve over another on the same paper (625). Colored inks (626). Incomplete figures (627).

Surprises. Questions are asked as to what will happen when the pen is compelled to draw two circles with equal radii and with equal or unequal periods, in the same or in opposite directions, and the like.

CHAPTER VII

STEREOSCOPIC HARMONIC CURVES

That harmonic curves* should be beautiful says that they have a quality, which, while it delights, does really only fulfil our expectations. But that they may be made stereoscopic, three-dimensional, standing out distinctly in space like woven wire forms, must come as a great surprise to those that have never seen them. And the method of drawing them is as simple as it is surprising. For this the writer is indebted to Pere DECHEVRENS.

711. THE PRINCIPLE. The two images, which the same solid object, provided it is not too distant, forms in our two eyes, are plane and

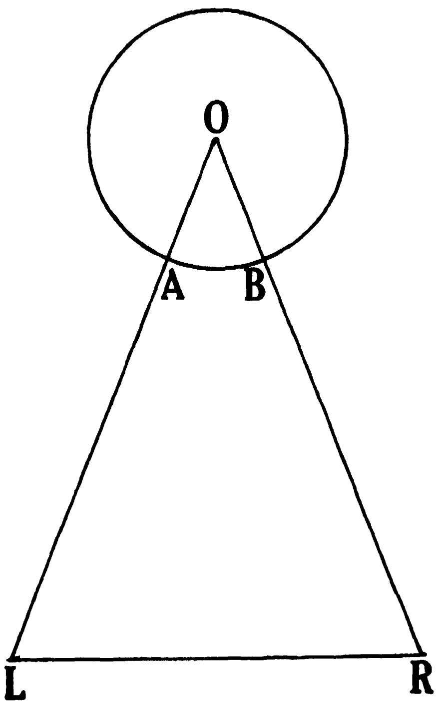


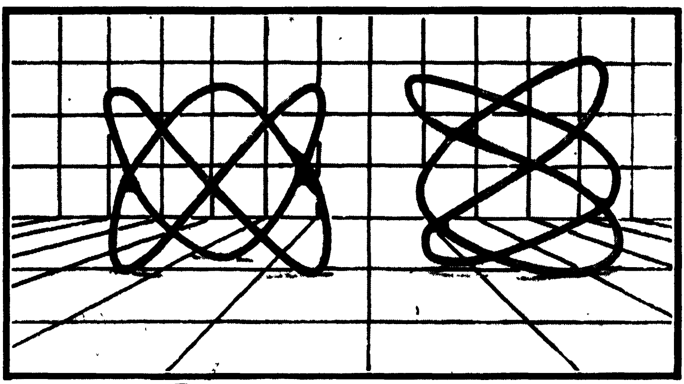
FIG. 711. Parallax, or Lateral Disparity.

* The matter of this chapter has been rewritten from an article with the same title which appeared in *School Science and Mathematics,* XXIV 29-36, in January, 1924.

slightly different. Thus when the eyes are directed towards a cylinder, Fig. 711, whose axis 0 is at right angles to the plane passing through it and the eyes L and R, the left eye L sees the point A apparently on this axis 0, while the right eye sees the point B in this position. The two images of the cylinder in our two eyes are therefore different. Although each is a plane picture, they both coalesce to form a solid one.

It is evident that if the cylinder is turned in the proper direction through the angle LOR, through the parallactic angle, as it is called, or the lateral disparity, the picture that was in one eye will be transferred into the other. This fundamental fact is the underlying principle of stereoscopic harmonic curves, in which this turning of the object through an angle is brought about by a change of initial phase.

712. Pictures and diagrams are essentially plane or two-dimensional. When they are intended to represent solids, lines in space, or three-dimensional objects, recourse must be had to some artifice in order to apparently lift them out of their "Flatland" of two dimensions into the



A B

FIG. 712. Wire Forms.

space of three dimensions. How hard this is every student of solid geometry knows. Modern text books, however, make this feat comparatively easy, by proper shading, suggestive lines, and especially by photographs, or even by stereoscopic views. Thus in Fig. 712, in which A and B are photographs of two different wire forms, while each is really a plane picture in two dimensions only, our imagination unconsciously and from an acquired habit at once supplies the third dimension and sees the original wire form. But when a piece of paper is held over one or both of these photographs and the wire form is carefully

traced on it, it is practically impossible for most people to even imagine this third dimension and see a wire form in the drawing.

713. Fig. 713E is such a drawing of Fig. 712A. When the wire form 712A is now turned towards the right through 15 degrees, its drawing or shadow or projection would be 713D. Turning it successively through 15° each time, would give Figs. C, B, A, respectively.

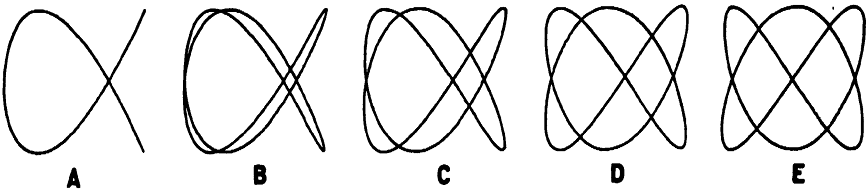


FIG. 713. Ratio 3:2.

714. *How to Look at Stereoscopic Pictures.* The five diagrams of Fig. 713 are thus five views of the same object Fig. 712A as seen by one eye when this object is turned through 15° intervals. All that is necessary now to get the stereoscopic effect and really see this wire form in its three dimensions, is to look at any one of these diagrams with the right eye only, and at the one adjoining it to the left with the left eye only. To do this, inexperienced persons must employ an ordinary stereoscope. This however can be used only when the distance between the centers of the pictures is equal to that between the eyes, that is, about 2¾ inches or 7 centimeters. When such an instrument is not available, the common way is to hold a card vertically between the two diagrams, taking care however to have both equally illuminated, so that the shadow of the card does not darken one of them. Greater practice will enable one to look at any pair of stereoscopic pictures without any instrumental aid whatever, and by directing the axes of the eyes to the middle of the pictures, obtain the three-dimensional effect in a few moments. This is not easy, but it is an accomplishment worth striving for. A preliminary lesson to it would be to place a small conspicuous object in such a position that each eye singly would see it in line with the middle of its picture either just above or through its center when the pictures are turned down. Both eyes are then directed to the object, and from it to the pictures when these are turned up.

715. Another difficulty will now threaten to make success impossible, and that is focussing our eyes on the pictures, which are nearer than

the object. Continuous practice has made us associate the converging of the eyes with their accommodation for a near object. These two operations are however not necessarily connected.* It is a feat worth the effort, and will be appreciated by its possessor in proportion to the labor it has cost him. So much so that the first time he really succeeds in this new optical experiment, he will experience a pleasure that must come upon a person who has had the use of only one eye all his life and is now suddenly blessed with the use of two. Two eyes give at once the relative distances of objects and of their parts, and make us see them as solid or three-dimensional. One eye can never do this in principle, because it can see only two dimensions, and the projections only of solid objects. It can, however, do it to a great extent in practice only because it is guided by experience and reason and imagination.

716. It may be of service to mention the fact that when the eyes turn from a distant object to look at a pair of stereoscopic pictures, each eye sees at first both pictures, so that there are four in all. By muscular effort the nearer two of these must be made to coalesce. There are then three, but attention must be centered on the middle of these three, which will then be three-dimensional. This imposes no more strain on the eyes than ordinary reading.

717. The diagrams in Fig. 717 are identical with those in Fig. 713 except that they are turned at right angles. They are the projections

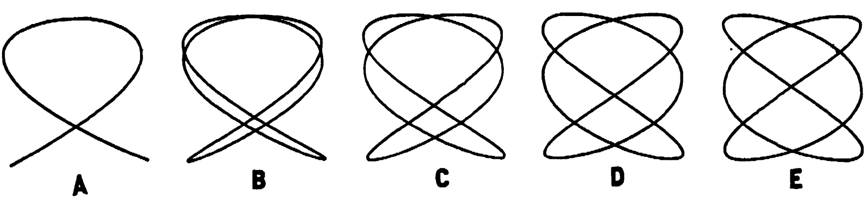


FIG. 717. Ratio 2:3.

of the wire form in 712 B, which may be called the reciprocal of 712 A, because the ratio of the cycles of the Y to the X component in A is $3/2$ while that in B is $2/3$.

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*The word "focusing" is often incorrectly used in this process. Focusing really means getting a distinct image, as in a telescope or microscope. It does not mean converging the axes of the eyes upon a given point.

718. Fig. 718 A and B are the developments of these two wire forms. When drawn on transparent material and bent into cylinders, they

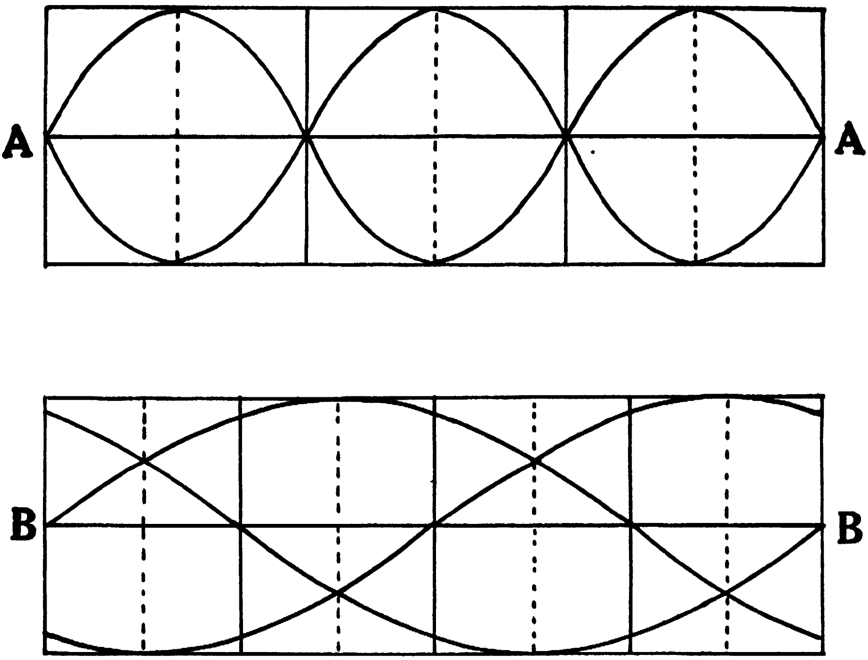


FIG. 718. The Wire Form Developed.

would form 712A and B perfectly and may be used in place of the wire. The shadows of these curves in Fig. 718 when so bent, as well as of the wire forms in 712, can be made to fit perfectly any one of the diagrams in Figs. 713 and 717.

719. The drawings in Figs. 713 and 717 are so simple that they may, in default of a machine, be drawn by points according to the method of Chapter III. The pleasure of the stereoscopic effect will repay a hundredfold the labor of making them.

721. *Merely a Change of Initial Phase.* As the simple secret of stereoscopic harmonic curves is merely a change in the initial phase of one or more of its components, their construction is as easy as their enjoyment is great. To make this change of phase clear, it will be well to look once more at Fig. 713 E. In this the *Y* component had 3 cycles while *X* had 2. The first started with phase 90° and the second with 0°, the starting point being at the top of the central lobe. In the

adjoining diagrams of Fig. 713 X started with 15°, 30°, 45°, 60°, respectively, while Y always began with 90°. When there are only two components, one in Y and one in X, as is the case here, and in fact even when there are several in both, it makes no difference in practice which is dephased. In principle however the phase of X must be changed, because that, as Fig. 711 shows, turns the object about the axis O, this being at right angles to the axes of the eyes. The change need not be large, five degrees or even three degrees often giving excellent results. Thus in Figs. 741, 743 D, E, which are triplex, Y having two components and X one, and in Fig. 742 which is quadruplex, both Y and X having two, the change was only 3 or 4 degrees. When X has two or more components, their changes of phase must however be proportional (966).

722. *The Distance between the Centers of the Pictures* must not exceed that between our eyes. It may be less, but it can never be greater, for the reason that the axes of our eyes have the power to converge or to be parallel, but not to diverge. When therefore the distance mentioned exceeds the assigned limit, we may either use a specially devised instrument with two or four plane mirrors (731,733), or else by means of some temporary intermediate point like that of a pencil, upon which we converge our eyes, look at the left picture with our right eye and at the right picture with our left eye.

731. *Stereoscopes.* Before proceeding further, it will be in place to mention a few stereoscopes. The instrument in which two plane mir-

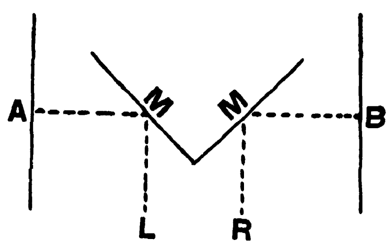


FIG. 731. Brewster's Stereoscope.

rors are employed is called BREWSTER'S stereoscope, which used to be mentioned in the old text books on physics. Two mirrors M are fastened at right angles on a board, Fig. 731. The stereoscopic pictures are placed at A and B, while the eyes are at L and R.

732. This form could not compete with the prism stereoscope, Fig. 732, now universally used, in which the two views *A* and *B* are per-

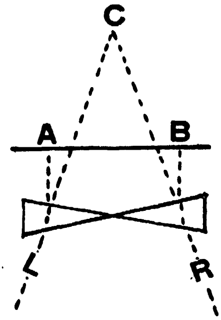


FIG. 732 The Ordinary Prism Stereoscope.

manently mounted on the same card, so that they not only cannot be separated, and go astray, but are also always perfectly adjusted except for distance, which is easily controlled. BREWSTER'S form has the further disadvantage that the harmonic drawings *A* and *B* must be secured by clips vertically in such a way that one or both may be rotated and moved up and down and sideways in order to adjust it to the other. The prism instrument, on the other hand, can be used only when the distance *AB* in Fig. 732 is equal to *LR*.

733. When four mirrors are used all these objections disappear. The eyes being at *L* and *R* in Fig. 733, the drawings are placed horizontally

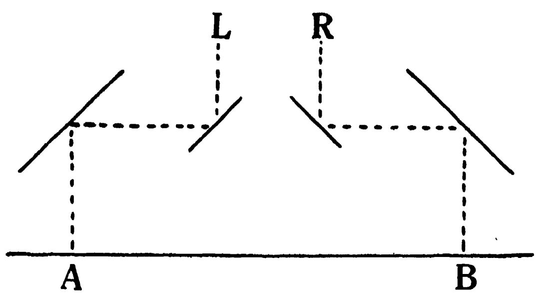


FIG. 733. The Four-Mirror Stereoscope.

on a board at *A* and *B*. They need not be fastened in any way, but simply laid down. They may then be adjusted very quickly by moving them about on the board. A good distance to use for *AB* is about 8 or 10 inches. In principle it may be anything. The heights of the mirrors may be adjustable or fixed.

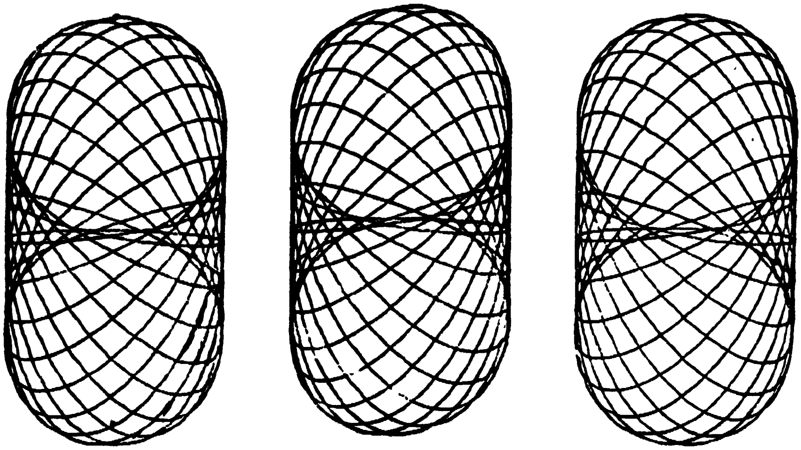


FIG. 741. A Wire Basket.

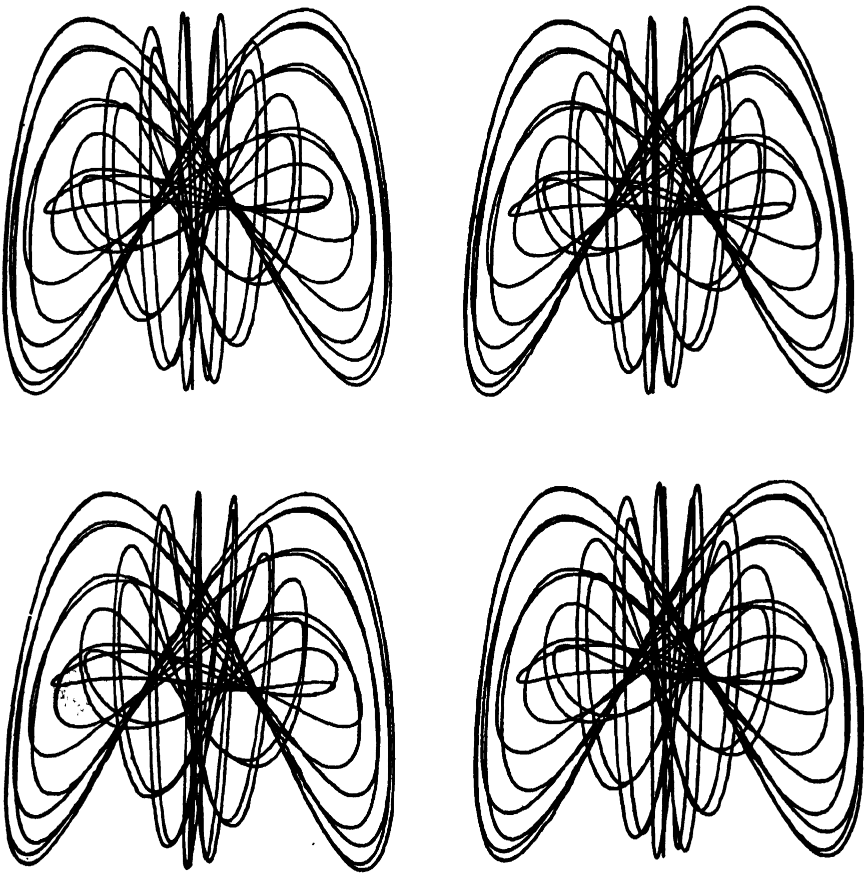


FIG. 742. Figures to be Turned at Right Angles.

741. *Exchanging the Drawings.* Those that are familiar with stereoscopic figures know that when the left and right pictures are exchanged, foreground and background also exchange places. Let the student practice on Fig. 741. Here the outer figures are identical, but the middle one has another initial phase. By this means the left pair differs from the right pair in that the right and left figures have been exchanged in them. It will be an agreeable surprise to him to see the wire basket turn its top towards him in one pair and its bottom in the other. It will not do to turn a single pair of diagrams like those in Fig. 743 halfway round, because while this exchanges right and left, it exchanges up and down also, so that the effect is lost and the same solid seen in both positions.

742. In Fig. 742 the upper pair differs from the lower one in that the right and left figures are exchanged in them, so that the diagonal ones are identical. The effect is the same as in Fig. 741, foreground and background being exchanged. Four figures are used for this purpose

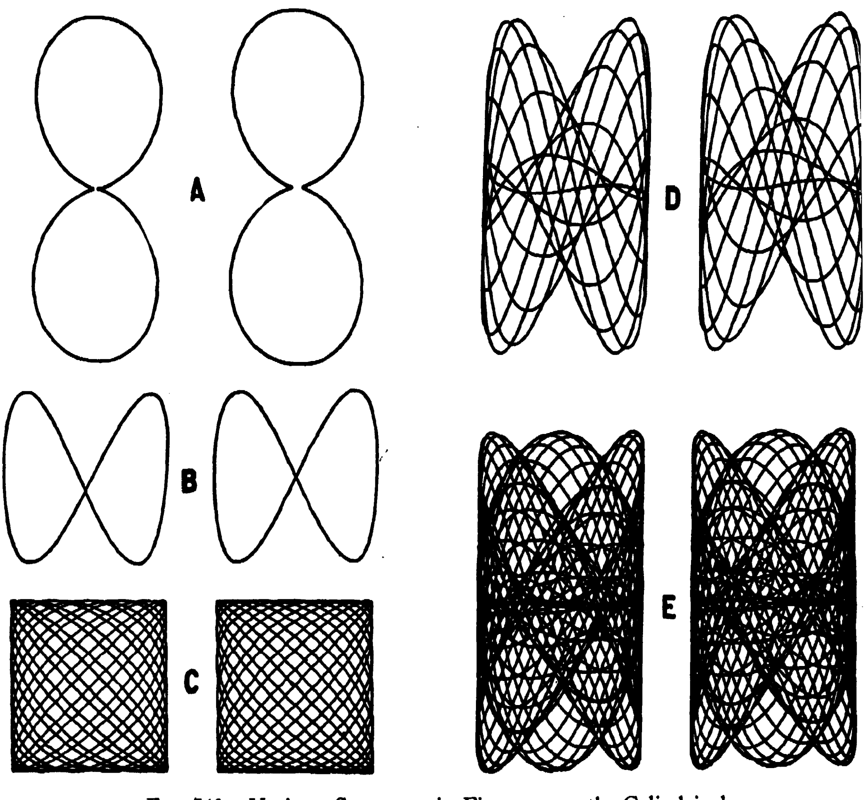


FIG. 743. Various Stereoscopic Figures, mostly Cylindrical.

in 742 instead of only three as in 741. This addition of the fourth figure is done however for another purpose. Let the reader turn the page at right angles and he will see a new three-dimensional form entirely different from what he saw before. The new upper and lower pairs exchange their new foreground and background as usual. The effect shown in Fig. 742 succeeds better, when, as is the case here, the figure is quadruplex, with two components in *X* and in *Y*.

743. In Fig. 743, B, C, D, E are four pairs of rectangular figures (A being polar) of increasing complexity. B is the curve mentioned in 224. C has 14 cycles in *Y* to 13 in *X*. It is a wire basket like Fig. 741, but seen on a level with the eyes. D and E are triplex with two components in *Y* and one in *X*. When *X* has only one component, the figures are always decidedly cylindrical.

751. *Polar Curves.* The remaining figures are all polar. Fig. 743A like 743B surprises one by its simplicity. 751A is also very simple. B has 11 cycles in *Y* and 12 in *X* in one rotation of the disk. C has 70, 42, 5 cycles respectively.. In 752 A is an epicycloid and B a hypo-

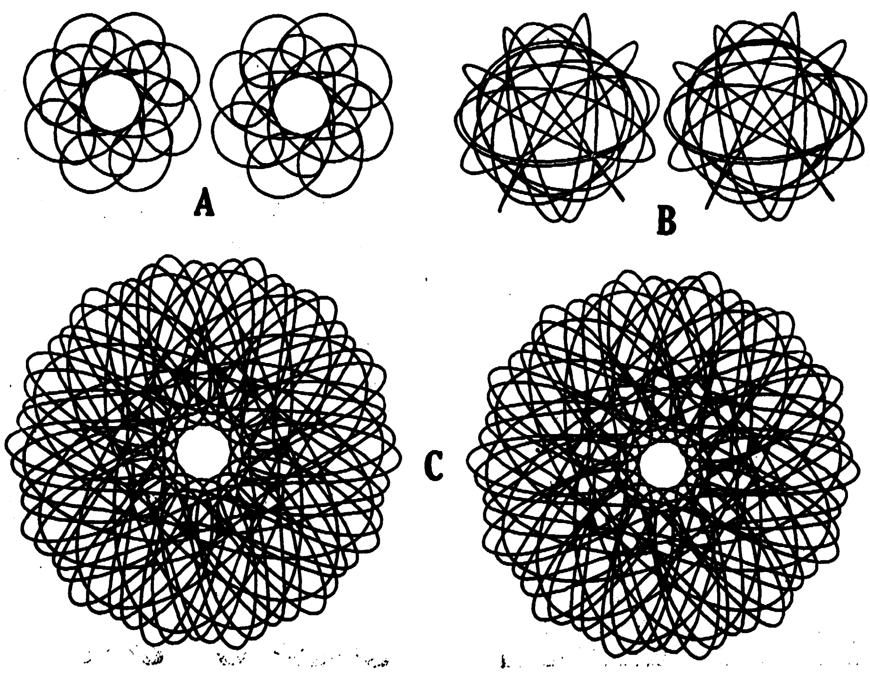


FIG. 751. Polar Stereoscopic Figures.

cycloid, both prolate (526). In 753A *Y* and *X* had each two components with cycles as 3:2; *Y* however had 16 of them while *X* had 15, and the disk turned twice. In 753B *Y* was to *X* as 16:15, 7 of these cycles occurring in 15 revolutions of the disk.

752. To dephase a polar curve there are at least two methods. The first is to change the phase of one or more of the *X* and *Y* components, and then, because this operation may displace the pen, to bring the pen back accurately to its original position. This is the best in principle, but is apt to prove very laborious in practice. The second method may be somewhat wrong in principle, but is very easy in practice. This simply displaces the pen a millimeter or two in any direction. The effect is as wonderful as the method is simple. This has been used in all the polar curves here.

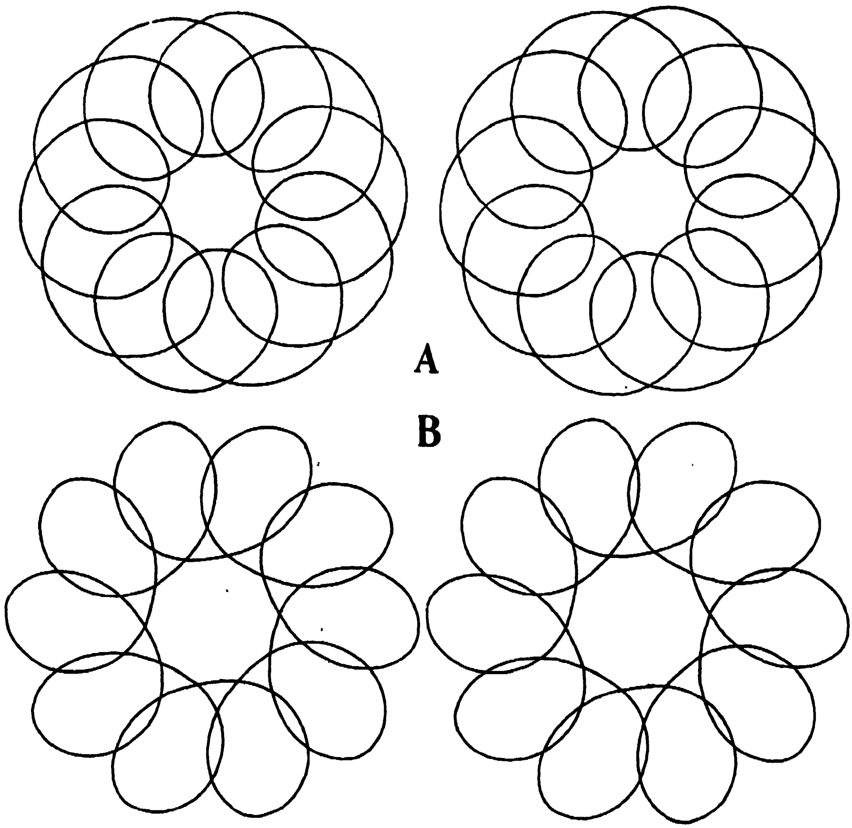


FIG. 752. Two Tori.

753. A peculiarity may be observed in Fig. 753B. This should be perfectly symmetrical, so that the wire ribbon should come from the bottom of every fold and go to the top of its neighboring one in the

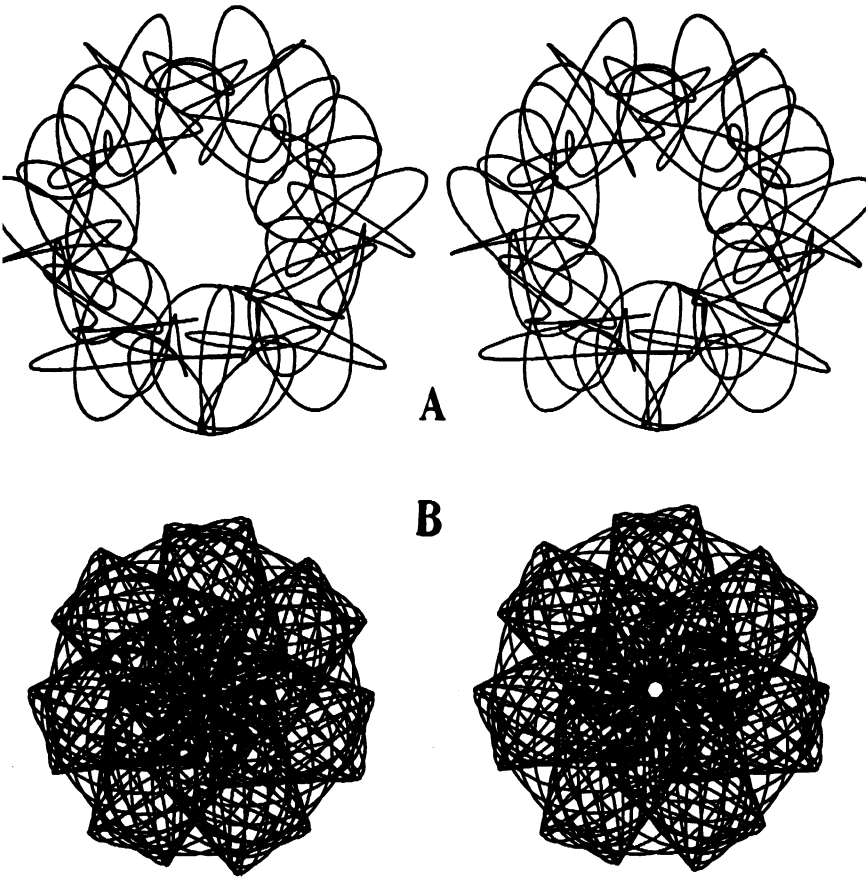


FIG. 753. Complex Stereoscopic Polar Figures.

same rotary direction. While this is true for the top and bottom folds, it is not so for the right and left ones, where it goes from bottom to bottom on one side and from top to top on the other. This anomaly, which passes unnoticed in most figures, may be very puzzling at first. The solution is that when the pen is displaced, it and the whole diagram are moved farther from or nearer to the center of the disk, as may be plainly seen in 753B, in which the right diagram has a larger central opening than the left one. Now, as the right eye sees the right figure only and the left eye the left figure only, when one looks at a corre-

sponding part to the right of the center, the axis of the left eye turns away from that of the right eye, because the right image is farther from the center towards the right, and hence the axes of the eyes converge less than they would do to see the paper, and therefore see their object farther away than the paper. Looking now to the left of the center, the axis of the left eye turns towards that of the right eye, the two eyes converge more and see their object nearer than the paper. Hence on the right side the ribbon of wire must run from the bottom of one fold to the bottom of the next, whereas on the left side it runs from top to top. The anomaly mentioned, once it is known, may be observed in many polar curves, in which the right side will appear to be nearer than the left, or vice versa, as the two drawings happen to be placed.

SUMMARY OF CHAPTER VII.

(711) The principle of stereoscopic vision.
(714-716) How to look at stereoscopic pictures without instrumental aid.
(721) Stereoscopic curves differ only in initial phase.
(722) The distance between the centers of the pictures.
(731-733) Three forms of stereoscopes.
(741-742) Exchanging the right and left drawings exchanges foreground and background.
(743) Examples of rectangular curves.
(751) Examples of polar curves. (752) How to dephase them.
(753) An anomaly explained.

CHAPTER VIII

ANALYSIS OF CURVES

811. While the direct object of this book was to show how to draw harmonic curves, that is, to synthetize them, according to the given amplitudes, periods, initial phases, and other parameters of their components, a concluding chapter on their analysis may be of much service, as it will enable the student to form some idea of the reverse and much greater problem, how from its mere figure, not only the number of the components of a complex curve, but all their parameters as well, may be determined with accuracy and even with rapidity. This problem, of course, called for a mathematical genius of a high order to blaze the trail, so that, while a thorough understanding of its solution may not be possible to all that draw harmonic curves, there are some main features of it easy to grasp. And it is the object of this chapter to present them.

812. The beginning will be made with a complex sine curve. This is a periodic single-valued function. It is periodic, in that a certain part of the curve repeats itself indefinitely in the same order and with the identical shape, just like what is called a repetend in an endless decimal. And it is single-valued, in that for every one value of x there is but one value of y. Curves of temperature, of heights of tides, and the like, are of this class. They are periodic, because they repeat themselves in the same order in a cycle of one or more years, and they are single-valued, because the temperature and the height of the tide at a given place have but one value at any one time.

813. FOURIER proved about a century ago (1822) that the equation of such a periodic curve could be expressed in terms of the sine and cosine of a variable angle x, plus the sine and cosine of $2x$, of $3x$, $4x$, and so on, with proper coefficients. In its rigorous mathematical form the equation is

$$y = \frac{1}{l}\int_0^l y\,dx + \left\{ \begin{array}{l} \left(\frac{2}{l}\int_0^l y \sin\frac{2\pi x}{l}\,dx\right)\sin\frac{2\pi x}{l} + \\ \left(\frac{2}{l}\int_0^l y \sin\frac{4\pi x}{l}\,dx\right)\sin\frac{4\pi x}{l} + \ldots \\ +\left(\frac{2}{l}\int_0^l y \cos\frac{2\pi x}{l}\,dx\right)\cos\frac{2\pi x}{l} + \\ \left(\frac{2}{l}\int_0^l y \cos\frac{4\pi x}{l}\,dx\right)\cos\frac{4\pi x}{l} + \ldots \end{array} \right.$$

For the purpose in view it is sufficient to call attention to the characteristics of this equation. It is seen to consist of pairs of terms which differ only in this, that sine and cosine are exchanged. Then, while the first terms have $2\pi x$, the second have $2 \cdot 2\pi x$ or $4\pi x$, the third would have $3 \cdot 2\pi x$, and so on. And lastly the l is the fundamental wave length, the abscissal length of the part of the curve that repeats itself, or the length of the first component, which has a period of one or unity. This is, as a rule, not difficult to measure. The problem in a given curve then consists in evaluating the coefficients enclosed in the parentheses.

814. The FOURIER equation may be written in simpler terms.

$$y = \beta + \left\{ \begin{array}{l} a \sin\theta + c \sin 2\theta + e \sin 3\theta + \ldots . \\ b \cos\theta + d \cos 2\theta + f \cos 3\theta + \ldots . \end{array} \right.$$

The chief characteristics mentioned before may now be seen more plainly.* The term β may be eliminated by shifting the origin. Its presence shows that the ordinates are not measured from the true axis of the curve. As the terms consist of sines and cosines of the same angles, and as these angles increase in a simple arithmetical progression, it is seen that the coefficients are really the only unknown quantities to be evaluated. There are various types of machines for doing this. As the HENRICI harmonic analyser is probably the least difficult to understand, a description of it will be taken from "The Science of Musical Sounds" by D. C. MILLER, published by The Macmillan Company in 1916. After quoting MILLER's own words, a detailed explana-

*Fig. 247 shows two such curves, one with only an 8th and 9th term, and the other with only a 4th, 5th, 6th, and 8th. It is well to emphasize the fact to one not familiar with it, that a Fourier series may consist of only a limited number of terms with large or small or no vacancies before or between them.

tion will be given of the operation of the machine and its solution of a practical example.

THE HENRICI HARMONIC ANALYSER

321. This instrument "devised by Prof. HENRICI, of London, in 1894, based on the rolling sphere integrator, is perhaps the most precise and convenient ever made. . . . The curve to be analysed, which must be drawn to a specified scale" [one compound wave length being equal to the width of the machine], "is placed underneath the machine; the handles *h* are grasped with the fingers, and the stylus *s* is caused to trace the curve, which requires movements in two directions. The machine as a whole rests on rollers which permit it to be moved to and from the operator, in the direction of the amplitude of the curve, and the stylus is attached to a carriage which rolls along a transverse track *t* in the direction of the length of the curve.

"The instrument shown [in Fig. 74] has five integrators; each sphere, made of glass, rests on a roller so that when the curve is traced, the sphere is rotated on a horizontal axis by an amount proportional to the amplitude of the curve; two integrating cylinders [with horizontal axes] with dial indexes rest against each sphere at points 90° apart, Fig. 75, and, by means of a wire and pulley *w* are given rotation about a vertical axis proportional to the movement along the axis of the curve. While each sphere rolls only in amplitude, the cylinders sliding around the sphere take up components of the amplitude motion which are proportional to the sine and cosine of the phase change respectively. The first integrator turns once around its sphere while the tracer moves over one wave length of the fundamental curve, that is, while the stylus is being moved the length of the track *t*, the next integrator turns twice, and the others, three, four, and five times in the same interval. In this manner one tracing gives the ten coefficients, five sines, and five cosines, of the first ten terms of the complete FOURIER equation of the curve.

"'In the Henrici analyzer the sizes of the various parts are so proportioned that the effects of the constant factors of the amplitude terms are mechanically incorporated in the dial readings, which are, without reduction (except for the factor *n* mentioned below), the actual amplitudes in millimeters of the components of the curve traced. When the stylus has been moved over one wave length of the fundamental, it must have moved over two wave lengths of the second component, three of the third, and so on; then the integrator for the second component has integrated two waves, and the dial readings are twice the required coefficients; in general, the readings of the *n*th integrator are *n* times too large . . .

"By changing the wire to the smallest pulleys v on the integrators, the spheres are turned six, seven, eight, nine, and ten times while tracing the wave, and the dials indicate the sine and cosine coefficients for the components from six to ten."

822. As an example MILLER then gives a curve which had only three components of measurable size, thus practically using only three sections of the machine and only one tracing. He gives the sine and cosine parts of each of the three components, and then combines each sine and corresponding cosine part into its own one component.

823. But there is one thing which MILLER does not explain, because it was obviously outside the scope of his book, and that is the theory of the HENRICI machine, how each sphere with its adjuncts does what he says it does. And this important missing part it is the intention to supply here.

THE THEORY OF THE HENRICI MACHINE

831. In order to begin with an easy case, let the curve have only one component, that is, let it be a simple sine curve, Fig. 831, so that only

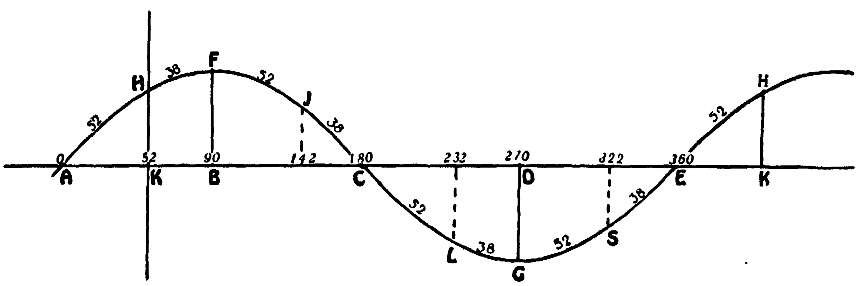


FIG. 831. A Simple Sine Curve to be Analyzed.

the first sphere in the HENRICI machine is needed. Let the sphere be viewed from above, and let the circle in Fig. 832 represent it. The roller, on which the sphere rests, is in the vertical plane through RT, and turns the sphere around the horizontal axis SN. It may be necessary to state that the sphere is held in place by its lowermost point where it touches the roller, and by a cup-shaped cap placed above it, and that the only function of the roller is to prevent the sphere from running away when the machine is lifted off the paper. There is no mechanical axis SN such as is used on terrestrial and celestial globes. The axis SN is purely mathematical and always parallel to the X axis

AE of the curve. When the stylus moves upward on the curve so as to increase the Y co-ordinate positively, the point R of the sphere also moves upward from the paper.

832. Let the curve start with the initial phase $\xi = 52°$, so that KH, the intercept on the Y axis, is equal to sin 52°, the amplitude BF being

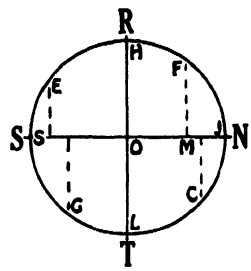


FIG. 832. A Sphere in Henrici's Analyser.

unity. Both the initial phase of 52° and the unit amplitude are here supposed to be known for the sake of a definite example; and it is the intention to show how the machine finds them. Otherwise they are, of course, not known at all.

833. Let the tracer be set at H on the curve, and let there be for the present only one integrating cylinder on the sphere, and let that be at R. As the stylus is moved along the curve, upward from H towards F, the R point of the sphere rises from the paper, and so does the sphere side of the cylinder. Let this direction of rotation of the cylinder be called positive, that is, when its sphere side rises from the paper.

The stylus rises at H, the sphere turns in consequence, and as R is on its equator, it turns rapidly and positively. The mechanism now swings R about 0 proportionally to the abscissa distance of the tracer from the Y axis, so that when the tracer is at F, R has been swung through KB/AE of 360°, = 38°. It is then at F in Fig. 832, where its rotary speed on its own axis is diminishing, since it is only as the distance F from SN, or as the sine of FON.

As the stylus descends from F in 831, the cylinder in 832 turns negatively and more and more slowly. It stops and begins to turn positively as it passes N, the "north pole" with the tracer at J at 142° in 831. The positive rotation continues until the tracer is at G in 831 and the cylinder at G in 832, 38° beyond the equator T. Then as the tracer rises from G in 831 the cylinder turns negatively, stops and turns posi-

tively as it passes S, the "south pole," with the stylus at 322°. Between S and H, the starting point, the rotation remains positive.

During one cycle, therefore, the cylinder in this case turns positively while the tracer moves from 322° to 90° and from 142° to 270°, and negatively the rest of the time. The positive rotation embraces $90° - 322° = 90° - -38° = 128°$, and $270° - 142° = 128°$, or in all 256°, and the negative rotation 104°.

834. The cylinder just followed is called the sine cylinder for a reason that will appear later on. While the sine cylinder starts at the equator R of the sphere, a second cylinder, called the cosine cylinder, starts at the north pole N and is always 90° in advance of the first. Its positive and negative rotations or readings may be studied in the same way as in that just given for its companion cylinder. There is, however, a simpler method, and this is, as said before, to call the rotation of the sphere positive when its R point rises from the paper, and the rotation of the cylinders positive when their sphere sides rise. As these are to be multiplied together, the sign of the product, that is, of the registration, is found very easily.

835. The fractional parts of the cycle during which the cylinders rotated positively and negatively are, however, not proportional to the linear distances their peripheries travelled, that is, to their readings or registrations, because the cylinders touched different parts of the sphere.

In order to get some idea of what these readings must be, the rollings of the sphere and of the cylinders must be studied somewhat in detail. When the stylus ascends from H to F in 831, it moves over $FB\text{-}HK = \sin 90° - \sin 52° = 1.000 - 0.7880 = 0.2120$. (See the table for the Sine Component in 838.) If the cylinder remained on the equator of the sphere at R, it would register this 0.2120. But during the interval it swings from H to F in 832, through $90° - 52° = 38°$, so that only the mean cosine of HF or of from 0° to 38° takes effect. To find this mean cosine, that is, the mean or average of all the cosines between HO and FM in 832, the method of the calculus is to find the area of the semi-segment $HOMF$, and then to put this area in a rectangular form with the same base OM, when its altitude will be the mean height of the arc or the mean cosine required.

The area of the semi-segment $HOMF$ is, if $x = OM = \sin 38°$, and $HO =$ unity,

$$\begin{aligned}
&\tfrac{1}{2}(x\sqrt{1-x^2} + \sin^{-1} x) = \tfrac{1}{2}(\sin 38° \cos 38° + \text{arc } 38°)\\
&= \tfrac{1}{4}\sin 2\cdot 38° + \tfrac{1}{2}\text{arc } 38° = \tfrac{1}{4}\sin 76° + \tfrac{1}{2}\text{arc } 38°\\
&= 0.2426 + 0.3316 = 0.5742.
\end{aligned}$$

Dividing this area by OM or sin 38°, gives 0.5742/0.6157 = 0.9328, as the mean cosine. Multiplying this by 0.2120, the distance the sphere rolled, gives + 0.1977 as the reading of the sine cylinder while the stylus moved from H to F in 831. (See the Table for the Sine Component 838.)

836. As the tracer continues from F to J in 831, from 90° to 90° + 52° or 142°, and moves over sin 142° — sin 90° = + 0.6157 — 1.000 = — 0.3843, the cylinder swings from F to J in 832. The area $MFJ = HOJ - HOMF = \frac{1}{4}\pi - 0.5742 = 0.7854 - 0.5742 = 0.2112$. This divided by $MJ = 1 - \sin 38°$, gives 0.2112/0.3843 = 0.5495, as the mean height of the arc FJ in 832. Multiplying this by — 0.3843 gives — 0.2112 as the cylinder motion (838).

837. Between J and C in 832 the area of the corresponding semi-segment is best found, as in the case just mentioned, by subtracting the area of the next section CL from a quadrant. To find the area CL the sine of 52° is now put in place of sin 38° in the section HF, so that as x is now sin 52° = 0.7880,

$$\begin{aligned}&\tfrac{1}{2}(x\sqrt{1-x^2} + \sin^{-1} x) = \tfrac{1}{2}(\sin 52° \cos 52° + \text{arc } 52°)\\&= \tfrac{1}{4} \sin 2 \cdot 52° + \tfrac{1}{2} \text{arc } 52° = \tfrac{1}{4} \sin 104° + \tfrac{1}{2} \text{arc } 52°\\&= 0.2426 + 0.4538 = 0.6964 = \text{area of } CL.\end{aligned}$$

Then 0.7854 — 0.6964 = 0.0890 = area of JC. Dividing this by 1 — sin 52° gives 0.0890/0.2120 = — 0.4199 as the mean height of the arc JC. And this multiplied by the rolling of the sphere, — 0.6157, gives + 0.2585 (see 838).

Dividing the area CL by sin 52° gives 0.6964/0.7880 = 0.8839 = mean height of arc LC. Multiplying this by sin 52°, in 831, gives back the area CL as the cylinder reading 0.6964.

The second half of the curve from L to H gives the identical results, as may be seen in the table. Adding up the cylinder readings, they are found to be + 1.8828 for the Sine Component.

The Cosine Component is now found very quickly. When the tracer starts at H in 831, the cosine cylinder is at N in 832. The increase of the ordinate is the same as before, and the mean heights of the arcs on the sphere are also the same, except that they are comparatively displaced. (See Table 838.)

838.

Section	Phase °	Increase of Ordinate of Curve	SINE COMPONENT: Mean height of Arc on Sphere	SINE COMPONENT: Increase of Cylinder Reading	COSINE COMPONENT: Mean height of Arc on Sphere	COSINE COMPONENT: Increase of Cylinder Reading
H	52	+ 0.2120	+ 0.9328	+ 0.1977	— 0.4199	— 0.0890
	38					
F	90	— 0.3843	+ 0.5495	— 0.2112	— 0.8839	+ 0.3397
	52					
J	142	— 0.6157	— 0.4199	+ 0.2585	— 0.9328	+ 0.5744
	38					
C	180	— 0.7880	— 0.8839	+ 0.6964	— 0.5495	+ 0.4330
	52					
L	232	— 0.2120	— 0.9328	+ 0.1977	+ 0.4199	— 0.0890
	38					
G	270	+ 0.3843	— 0.5495	— 0.2112	+ 0.8839	+ 0.3397
	52					
S	322	+ 0.6157	+ 0.4199	+ 0.2585	+ 0.9328	+ 0.5744
	38					
E	360	+ 0.7880	+ 0.8839	+ 0.6964	+ 0.5495	+ 0.4330
	52					
H	52	0.0000		+ 1.8828		+ 2.5162

839. Upon adding up the tabulated items for the sine and cosine components, the registration of the sine amounts to $a = + 1.8828$ and of the cosine $= b = + 2.5162$. The square root of the sum of these numbers squared is 3.143, which is remarkably close to $\pi = 3.142$, as it ought to be in principle, and as it occurs in the original FOURIER formula, the reason for which will appear presently. This is the amplitude desired, which ought to be unity.

It may be made so in three ways. First, this number may be regarded as an instrumental constant, and all the cylinder readings divided by it. Secondly, the diameters of the cylinders may be made 3.142 times as large as that of the sphere, so that one of their turns would be unity. Thirdly, and best of all, the cylinders may be made one-tenth as large as this, with diameters 0.3142 times that of the sphere, so that one turn will be one-tenth of unity. And this last is the method adopted in practice.

841. Dividing the sine and cosine readings, + 1.8828 and + 2.5162, by $\pi = 3.142$, makes $a = +0.5994$ and $b = + 0.8017$. The first terms in the Fourier series are then $+ 0.5994 \sin \theta$ and $+ 0.8017 \cos \theta$. The first is a sine curve, the dashed one in Fig. 841, and the second a cosine curve, the dotted one in the same figure. This second one may then be written $+ 0.8017 \sin (\theta + 90°)$, that is, with 90° as an initial phase.

These two curves in 841 are now to be combined into one by the method given in 322, 324. The amplitude is then $\sqrt{(a^2 + b^2)} = 1.002$, and the initial phase is the arc whose tangent is b/a, that is, $\xi = 53°13'$. In view of the many items that entered into their computation, these values are sufficiently close to the assumed one 1.000 and 52° (832). The resultant (full line) curve then has the equation $y = 1.000 \sin (\theta + 52°)$.

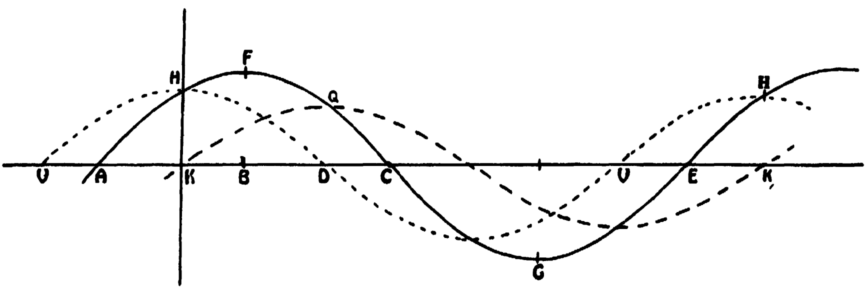


FIG. 841. Sine and Cosine Component Curves.

As the amplitude $FB = \sqrt{(a^2 + b^2)}$ is here 1.000 and $b/a = \tan \xi = \sin \xi / \cos \xi$, it follows that here $a = \cos \xi$ and $b = \sin \xi$. This is the case only when the amplitude is unity, so that the designation of sine curve and cosine curve does not come from a and b. Sine curve means that the initial phase is 0°, and cosine curve that it is 90°, or again, that the intercept on the Y axis is respectively sin 0° and cos 0°.

842. The essential principle of the HENRICI analyzer may be well seen in these component curves $a \sin \theta$ and $b \cos \theta$, which are dashed and dotted respectively in 841, and which have, of course, the same period or wave length as their resultant has, $KK = VV = AE$. Let the stylus trace the sine curve KK with the sine cylinder alone. It starts at K in phase 0°, and the cylinder is on the equator of the sphere at 0° at R in 832. As the stylus rises the height $DQ = a \sin 90° = a$, the cylinder swings from R to N. The mean height of the arc RN in a unit circle is $\frac{1}{4}\pi$, and therefore the cylinder record is $+\frac{1}{4}\pi a$. As the tracer descend from Q in phase 90° to phase 180° in 841, it moves over $-a$, and the cylinder swings from N to T in 832, the mean height of LJ being $-\frac{1}{4}\pi$, and hence the cylinder reading $+\frac{1}{4}\pi a$. This reading is the same in the third quadrant, and also in the fourth, so that the sine cylinder record is $+\pi a$. The reason for dividing the record by π(839,841) is now apparent, this division being done numerically or

mechanically as mentioned before. The sine cylinder therefore records the amplitude or the coefficient a of $a \sin \theta$.

843. When the stylus starts at H in 841 to move over the cosine curve with the cosine cylinder at the north pole N in 832, it descends through two quadrants through $2b$, while the cylinder remains on the negative half of the sphere, thus giving a plus record. During the third and fourth quadrants the stylus rises with its cylinders on the positive half of the sphere, so that the total record of the cosine cylinder is $+\pi b$, or simply $+b$, as said before. It is in this way that, when the stylus is moved over the curve to be analyzed, the sine and cosine cylinders record the amplitudes of the coefficients a and b of the component sine and cosine curves.

844. These two, a and b, then determine the amplitude $\sqrt{(a^2+b^2)}$ of the curve analyzed, as well as its starting phase ξ, the tangent of which is b/a. While, for the sake of simplicity, the amplitude of the curve which was taken as an example was 1.000, the principle would be everywhere the same if it was m times as large. But it would have been awkward to drag this factor m along through the whole investigation. And lastly, it is understood that b/a must be taken according to the signs of a and b, so that there be no mistake about the true value of ξ.

845. In the case illustrated the axis of the curve AE in 831 was supposed given. When this is not known, a line is drawn tangent to the lowermost points of the curve, as in Fig. 845, and used as a temporary X axis. The Y axis is then drawn through one of these points or through any other well marked one. The length of a wave is the distance between two similarly placed points, such as A and B in 845. With a planimeter the area $DACH + HLBE$ between the curve and the temporary X axis is measured. This area divided by the wave length DE is then the mean height DF of the curve, the height of the true X axis FG above the temporary one, and the term β in the FOURIER equation (814).

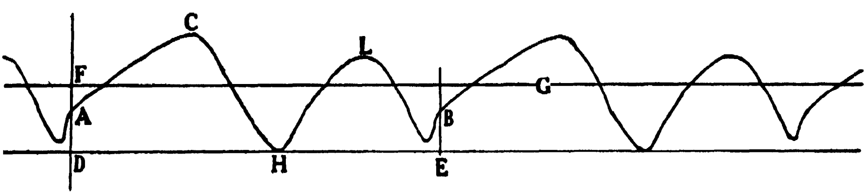


FIG. 845. An Unknown Curve.

When the tracer starts from the temporary X axis DE in 845 and runs up DA on the Y axis to the curve, the cylinders remain at R and N in 832. The sine cylinder is at the equator of the sphere and records the height DA in its full value, while the cosine cylinder N at the north pole does not rotate at all. When the full wave length of the curve AB has been gone over by the stylus, this last, in running down BE to the temporary X axis DE, then undoes completely the record DA of its ascent, because the points of the beginning A and end B of a wave length are evidently at the same height, $DA = BE$, and the cylinders are in their original positions.

846. In the same way if the ordinates of a curve should be measured from a slanting line, just as would be done in the case of oblique axes, and this straight slanting line should continue to ascend during one wave length of the curve and then descend equally during the second, whatever increase the cylinders would record during the first wave length, the very same would be subtracted during the second. It would be just as if there were no slanting line at all, so that the mean or the half of the two waves would be recorded properly.

The very same would be true if, instead of an ascending and descending straight line, there were a curve of one wave length, on which the two waves of the curve under investigation were superposed. The mean or the half of the readings would again be a correct record of it. The same may be said of 3, 4, or more waves superposed on a fundamental one of one wave length. Each set of cylinders would then rotate about its respective sphere 1, 2, 3, or more times with perfect independence, so that any or all of them may be used at pleasure, their readings being then divided by the number of their waves.

847. In this way when a curve is a sine curve, that is, begins with phase 0°, the sine cylinder alone will give the record (842), while the record of the cosine cylinder will be zero, as can readily be proved. In like manner for a cosine curve, which is a sine curve beginning with phase 90°, the cosine cylinder alone will record (843), while the sine cylinder will read zero. From this it follows that, because every simple curve may be taken as a compound of a sine and a cosine curve (324), the sine cylinder will record the sine component, and the cosine cylinder the cosine component, each acting with perfect independence of the other. And this is the principle of the HENRICI analyzer.

848. This shows also how such an analyzer with five spheres can in one journey of the stylus record the sine and cosine parts of each of

five components of the original curve. The number of spheres might be made greater than five, but it is found best in practice to move the stylus over the curve a second time and to swing the cylinders round 6, 7, 8, 9, 10 times. MILLER (p. 100) has reconstructed his HENRICI machine, so that in running its stylus over the curve six times it may record 30 components with precision.

"The verification of an analysis is made by synthesis," he says (p. 128). Hence when the components of a curve have been found with their amplitudes and initial phases—their periods being as the numbers 1, 2, 3, 4, . . . n,—these may be set up on a synthetizer and the compound curve drawn by machine. The scale having been made the same, the analyzed and synthetized curves may be superposed one on the other, and thus the correctness of the analysis verified. MILLER on p. 127 shows both curves for which 12 components were used. The agreement of the two is wonderfully close.

849. In the hands of an expert like MILLER the HENRICI machine works not only with precision but also with rapidity. A ten-component curve was analyzed by the machine (p. 136) in 13 minutes, while numerical methods required from about 3 to 10 hours. The synthetizer could be set for ten components and the curve drawn in 5 minutes (p. 121), while for 30 components only 12 minutes were required. This rapidity is mainly due to the fact that only the amplitudes and initial phases need be set, because the gearing for the periods always remains the same.

OTHER HARMONIC ANALYZERS

851. There are also other forms of harmonic analyzers. In MICHELSON's machine 80 components may be both analyzed and synthetized. For the first the curve must be cut on the edge of a card or a sheet of metal. In other machines the scale of the curve may be of any magnitude, but it must be traced by the stylus for each component separately after the gearing has been properly set for it.

852. The above must suffice for the purpose of this book. Further information may be obtained from MILLER's "The Science of Musical Sounds," and from "Modern Instruments and Methods of Calculation: A Handbook of the Napier Tercentenary Exhibition." Both of these book give abundant references to other works or articles.

853. Only this need be said in conclusion, because neither of the books just mentioned treats it. It is the application of this analysis to other

harmonic curves besides the complex sine curves. In polar curves in which the pen moved on a radial line, the same method could be used if the curve is turned about its center instead of advancing the analyzer parallel to the X axis. When the pen travels on a non-radial line in drawing the curve, the slant of this ought not to be difficult to find. When the pen has both X and Y components, these may probably be best found for one axis at a time. A turning disk would increase the complexity; closeness or overlapping of paths might make the solution hopeless. It is not likely that those who are fond of synthetizing harmonic curves will do much work at analyzing them.

Summary of Chapter VIII

(811 To draw a curve means to synthetize it, "to put it together," according to its given components with their amplitudes, periods, and initial phases. To analyze it is the converse problem, to ferret out from the given curve all its components with their parameters.

(813) The equation of a periodic single-valued function was proved by Fourier to consist of a series of terms arranged according to the sines and cosines of θ, 2θ and 3θ, and so on, with constant coefficients. Analyzing the curve is thus narrowed down to finding these coefficients.

(821) The Henrici harmonic analyzer is described.

(831-849) Its theory is explained to rest on the fact that the terms containing $\sin\theta$ and $\cos\theta$ (and similarly of $\sin 2\theta$, $\cos 2\theta$, etc.) represent the sine and cosine components of a curve of one period (and in like manner of 2, 3, etc. periods). The amplitude is the square root of the sum of the squares of these two corresponding coefficients, and the initial phase has as its tangent one divided by the other. The theory is illustrated by a practical example.

(851) Other harmonic analyzers are referred to.

CHAPTER IX

A FEW HARMONIC CURVES

In this chapter a few of the simpler curves, such as a straight line, a circle, an ellipse, and a rectangular curve, are shown in what may appear to many novel and unexpected conditions. The sections are prefaced by methods of drawing the simple curves themselves, beginning with the point.

I. Point

911. An instantaneous position of the pen in a curve, or a position in a curve when there is no motion, as in a cusp, gives a point with the co-ordinates x, y.

912. The position of the pen at the center of the rotating disk ($x=0$, $y=0$, $\varrho=0$) when either all the parameters are zero in 235, or the motions are equal and opposite as in 542, 642, etc.

913. The hypocycloid of one cusp (542), is a point.

II. Straight Line

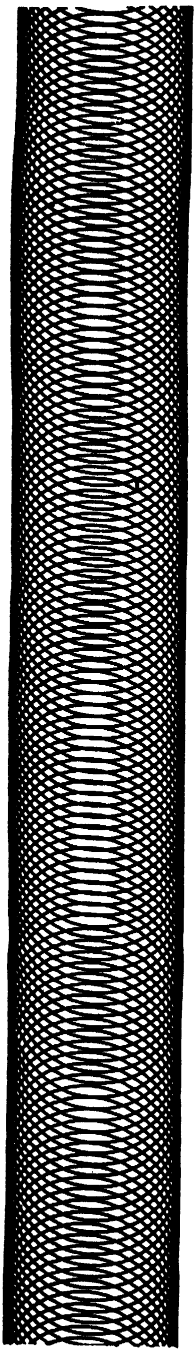
921. *A Constant Straight Line* is here defined to be the line traced by the pen when it has only one component, and the paper is stationary. It would be the vertical diameter of the circle in Fig. 211. The length of the line is then twice the amplitude. Its equation is $x=0$, or $x=\pm a$.

As said repeatedly, when the pen has one or more components in one axis only, this is always taken as the Y axis, and when the paper is moved, its motion is always at right angles to this Y axis.

The line is inclined when in a rectangular curve all the periods are equal and all the starting phases zero (or equal), so that (223)

$$x=(a+c+e+\ .\ .\ .)\sin\theta$$
$$y=(b+d+f+\ .\ .\ .)\sin\theta$$
$$\text{and } y=(b+d+f+\ .\ .\ .)\,x/(a+c+e+\ .\ .\ .).$$

The hypocycloid of two cusps (544) is a straight line, $x=2\cos\theta$, $y=0$.



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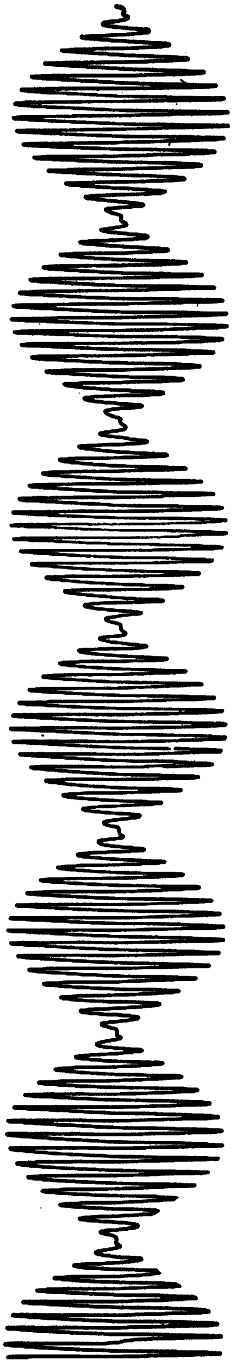

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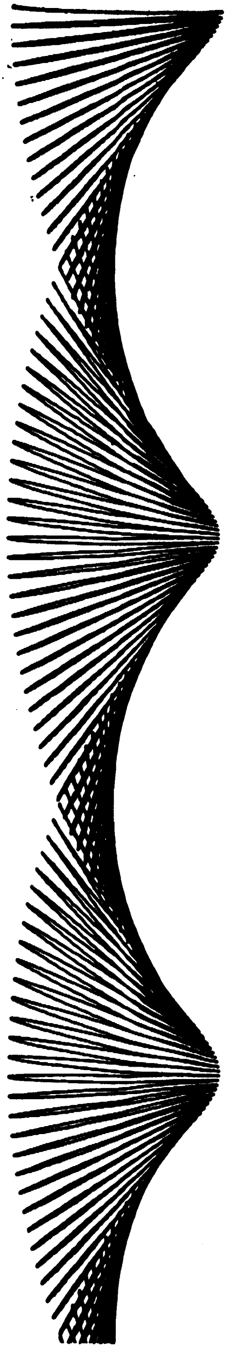

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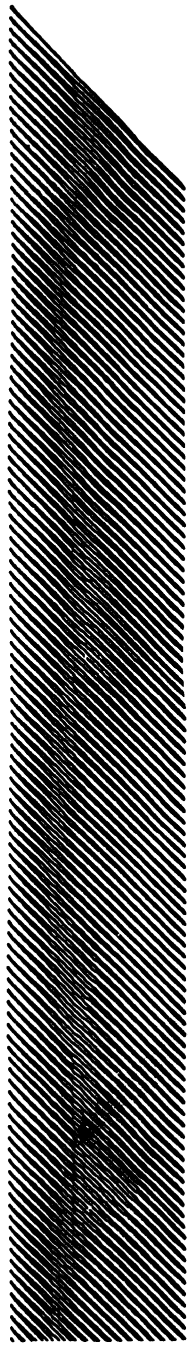

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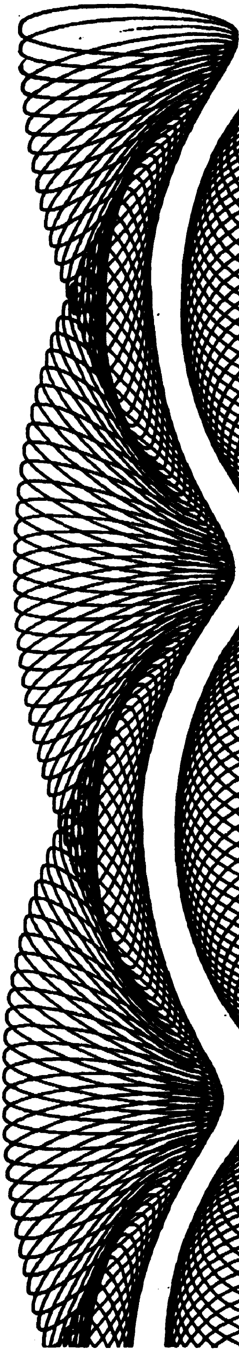

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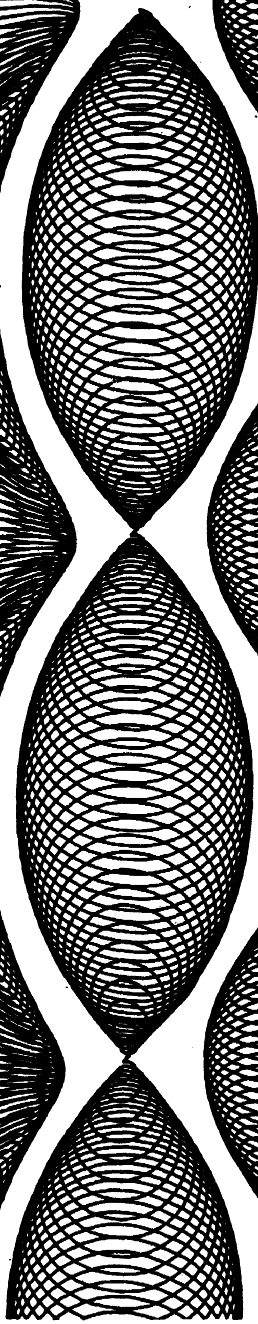

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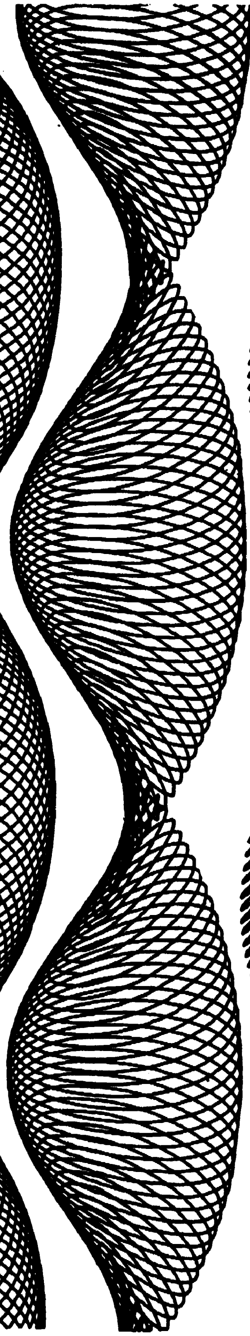


948

926 B

922. *A Constant Straight Line with its Midpoint moving in a Straight Line.* When the first straight line is in the Y axis and the second is the X axis, the resultant curve will be a simple Sine Curve (Figs. 211, 212).

It is understood, of course, that the sidewise shifting of the Line or of any Curve takes place while this Line or Curve is being drawn.

When the Constant Line is not in or parallel to the Y axis, as in Fig. 922, a slanting sine curve is produced,* just as if oblique axes were used.

$$x = a \sin \theta + \theta/47, \qquad y = b \sin \theta.$$

923. *A Constant Straight Line with its Midpoint moving in a Circle.* Fig. 923 gives the impression that while the line was being drawn, its

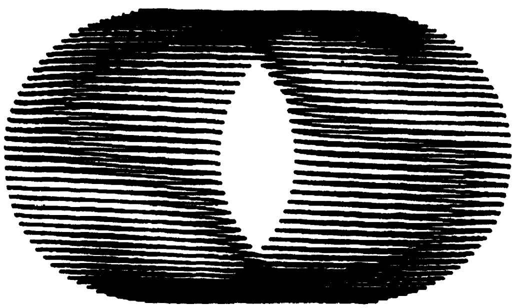


FIG. 923. A Constant Straight Line with its Midpoint Moving in a Circle.

direction remained constant, while its midpoint moved in a circle. The obvious way of doing this mechanically would be to have one component in Y for the line, and then a pair of circular ones (223) with a large amplitude and a very great period in both X and Y, with stationary paper.† The equation would then be

$$\begin{aligned} x &= \qquad\qquad 1.33 \cos \theta \\ y &= \sin 89\, \theta + 1.33 \sin \theta. \end{aligned}$$

But on account especially of the comparatively very slow rotation of the pen in the circle, this method could not be used on the CREIGHTON

* Owing to the length of the rectangular-sine curves, it was necessary to combine them in groups in order to save space. This somewhat disturbs their regular sequence and prevents placing them in their proper paragraphs (32).

† At first thought one might imagine that the figure could be drawn by having one component in Y in a radial line, and then rotating the disk slowly. This however would keep the line radial and make it turn on the paper 360° in one revolution. It would generate Figs. 924 and not 923.

machine. The one adopted was to draw the Rotating Straight Line of Fig. 924A (as in the following case) twice, so that the line swung through 360° (instead of 180°) and there were two complete complex cycles of the pen, and then to annul this swinging on the paper by turning the disk in the same direction and with the same angular speed. The equation is the same as in 239, except that the signs of the 5th and 7th terms of x are to be reversed and r made equal to 2.66.

924. *A Rotating Straight Line with Fixed Midpoint.* The pen is made to draw two circles with equal radii, in opposite directions, and with slightly unequal speeds (644). The equation is

$$x = \cos 44\,\theta + \cos 45\,\theta$$
$$y = \sin 44\,\theta - \sin 45\,\theta.$$

The effect is shown in Fig. 924A (in the right upper corner of the Frontispiece), in which the line appears to swing about its midpoint. The figure is in reality an equifoliated rosette with $45 + 44 = 89$ lobes, and might have been drawn as such. But its shape was not foreseen (644). As these lobes are very narrow and are bent very little out of their way, they look like straight lines.

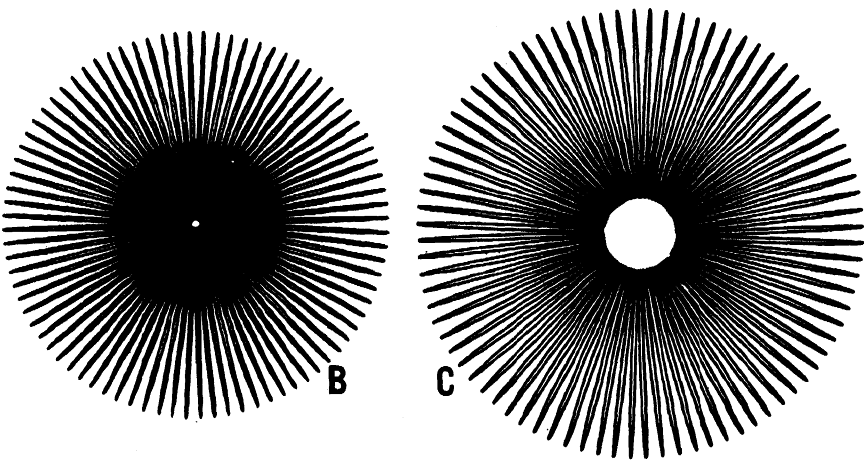


FIG. 924. A Constant Straight Line Rotating.

When the line rotates about an end point, the figure has the appearance as in 924B, where $\rho = 1 + \sin 84\,\theta$. When the point is outside of the line, but in its direction, as in C, where $\rho = 1.33 + \sin 84\,\theta$, there is a vacant circular spot in the middle. When the point is outside of the line, but not in its direction, the figure is skew, as in the Appendix, Cuspidal Rosettes, Fig. 24, $\rho^2 = (1 + \sin 84\,\theta)^2 + 0.45^2$.

A Rotating Straight Line with its Midpoint moving in the X Axis. Fig. 924D. The equation is the same as before (924A), with $g\theta = 10\theta$ added to x.

925. *A Variable Straight Line* The pen has two components in Y only with amplitudes a and b and different periods, over stationary paper. The length of the line varies between $2(a+b)$ and $2(a-b)$. The effect is best when $a = b$ and the periods have a close ratio such as 44 : 45. $y = \sin 44\theta + \sin 45\theta$.

The variable line may be given any inclination, when there are two sets of equal components in both axes, with equal starting phases, but with unequal periods, such that

$$x = a \sin 44\theta + a \sin 45\theta$$
$$y = b \sin 44\theta + b \sin 45\theta.$$

926. *A Variable Vertical Straight Line with its Midpoint moving in the* X *Axis,* is a Sine Curve with two components, Fig. 926A. The figure is best when the amplitudes are equal and the periods slightly unequal. The speed g may then be suitably chosen. The equation is (216)

$$x = 3.75\theta$$
$$y = \sin \theta + \sin {}^{15}\!/\!_{16}\theta.$$

A Variable Slanting Straight Line with its Midpoint moving in the X *Axis* generates Fig. 926B. In the equation $g\theta = 10\theta$ is added to x in 925.

927. *A Rotating Variable Line with Fixed Midpoint,* as in Fig. 927A, resembles two fan leaves or shells.

The envelope consists of two equal circles tangent to each other, and the rotating variable line always passes through the point of contact. The pen moved in a radial line. It had two equal components of periods 44 and 45. The disk made only half a turn in one compound period, and thereby completed the figure. The pen was started at the center in phases 0°. The equation is (242)

$$\varrho = \sin 2 \cdot 45\theta + \sin 2 \cdot 44\theta.$$

When the disk makes a whole turn in one compound cycle of the pen, so that

$$\varrho = \sin 45\theta + \sin 44\theta,$$

the envelope is two symmetrical partially-overlapping cardioids. Fig. B.

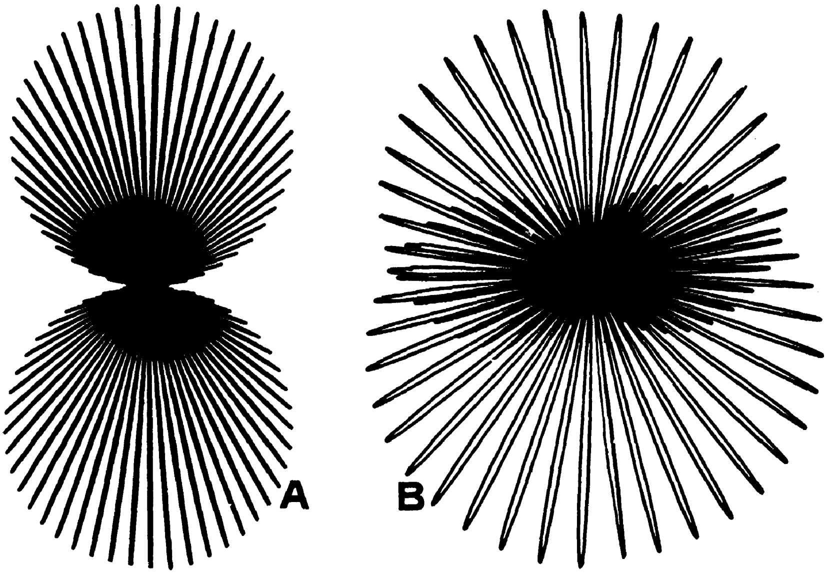


FIG. 927. A Variable Straight Line Rotating.

928. *A Rotating Variable Line with its Midpoint moving in the* X *Axis* was not drawn. It would look somewhat like 947, but not as fine.

929. *A Rotating Variable Line with its Midpoint moving in a Circle.* When the path of the pen is tangent to the circle, the equations are

$$\varrho^2 = (\sin 45\theta + \sin 44\theta)^2 + 4$$
$$\text{and } \varrho^2 = (\sin 45\theta + \sin 44\theta)^2 + 1$$

according as the radius of this circle is respectively 2 or 1. Figs. 929A and B are the results (B is in the left lower corner of the Frontispiece).

In Fig. C the pen moved on a radial line with its midpoint on a circle with radius 2. There were 2 cycles of the pen in one turn of the disk. Here again there are two symmetrical and overlapping cardioid envelopes, the common part however being vacant. (Compare 927B.) The equation is (242)

$$\varrho = 2 + \sin 2 \cdot 45\theta + \sin 2 \cdot 44\theta.$$

When the disk makes one turn only in 10 or 5 compound cycles of the pen, Figs. D and E show the results. The equations are the same as the one just given except that 10 and 5 respectively replace the factor 2 in θ.

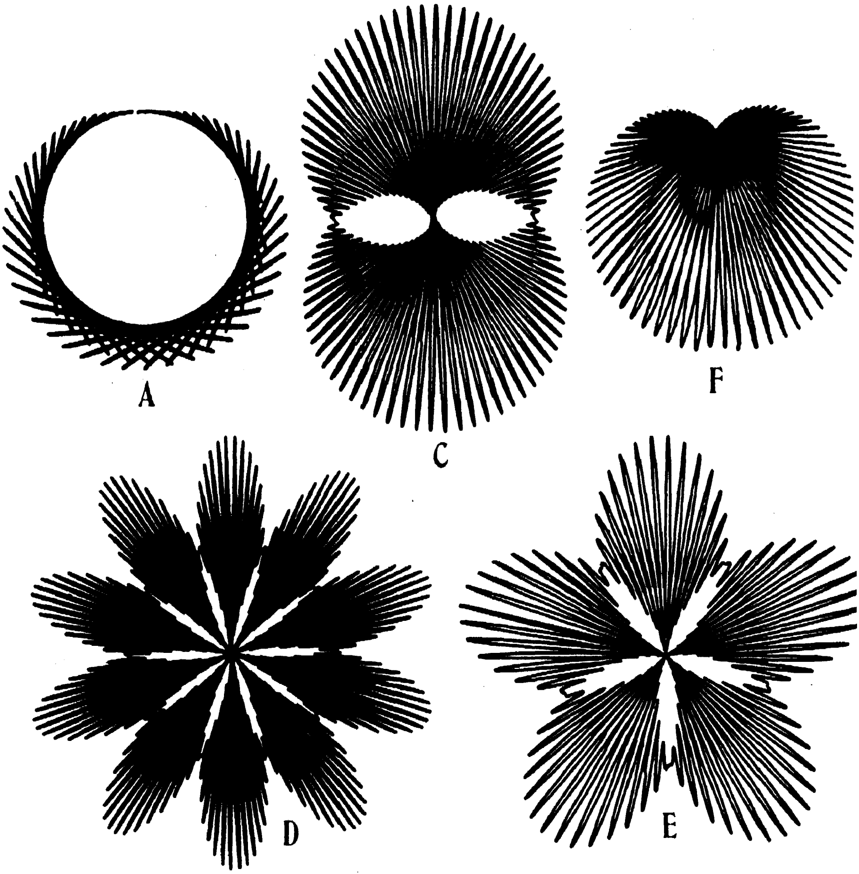


FIG. 929. Variable Straight Line Rotating.

Fig. F differs from C in two particulars. First, the pen was started in phases 90° at the distance 1 beyond the center of the disk, and secondly, there was only one compound cycle of the pen instead of two. The equation is

$$\rho = -1 + \sin 45\theta + \sin 44\theta.$$

The envelope is again a cardioid.

III. CIRCLE

931. A *Circle* is drawn when the pen is stationary and the disk revolves, that is, when in 233 a and b are zero, so that $x = \cos(\theta - \varphi)$, $y = \sin(\theta - \varphi)$, or $\rho = r$.

A circle is a rectangular curve (223)

$$x = a \sin(\theta + \xi), \qquad y = a \sin(\theta + \eta)$$

when $\xi - \eta = \pm 90°$. The rotation is direct or anti-clockwise when x leads in phase.

The circle is a rosette (244), $\rho = \beta + b \sin(n\theta + \eta)$, with $n = 1$, $\beta = 0$, so that $\rho = b \sin(\theta + \eta)$.

932. *A Circle with its Center moving in a Straight Line* generates a common cycloid, with its varieties of curtate or prolate, depending upon the value of g in the equation (513)

$$x = g\theta - a \sin\theta$$
$$y = a(1 - \cos\theta).$$

Fig. 932 is a prolate cycloid (253)

$$x = -\cos 48\theta + \theta$$
$$y = \sin 48\theta.$$

933. *A Circle with its Center moving in a Circle.* This is a circular cycloid (521). When the center of the rotating circle revolves in the same rotary direction as the tracing point, the curve is an epicycloid, when it revolves in the opposite direction, the curve is a hypocycloid. See Chapter V.

It is the property of a harmonic curve that the pen moves in a circle, the center of which moves in a second circle, and so on (233).

934. *A Variable Circle with Fixed Center.* A variable circle, that is, one with a variable radius, is produced when the pen is made to trace two circles with equal radii in the same direction but with unequal speed (642).

$$x = \cos 45\theta + \cos 44\theta$$
$$y = \sin 45\theta + \sin 44\theta.$$

Fig. 934 shows that the pen starts by drawing a circle with radius 2, and that this diminishes to zero in successive rounds, and grows again to 2. When the radii are unequal, there is a vacant circular space in the middle.

935. *A Variable Circle with its Center moving in a Straight Line.* The term $g\theta = 10\theta$ is added to x in the foregoing equation (934). Fig. 935 illustrates it.

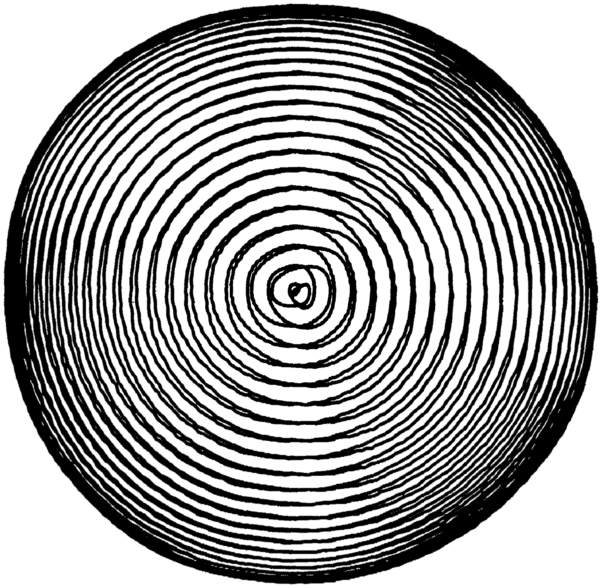


FIG. 934. A Stationary Variable Circle.

936. *A Variable Circle with its Center moving in a Circle.* Fig. 936 shows two compound cycles of the preceding curve 935 bent into a circle. Its equation is given in 239.

IV. ELLIPSE

941. The *Ellipse* is a rectangular curve (223)

$$x = a \sin(\theta + \xi), \quad y = b \sin \theta,$$

for all values of a, b, ξ. In the equation just given, when $a = b$ and

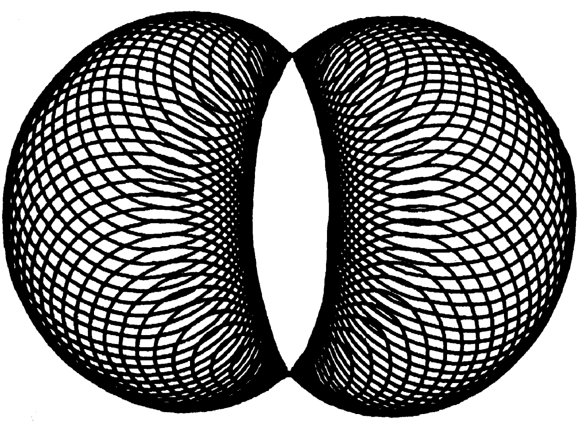


FIG. 936. A Variable Circle with its Center Moving in a Circle.

$\xi = \pm 90°$, the ellipse becomes a circle, and when $\xi = 0°$, it is a straight line. This is a rapid way of changing a circle, an ellipse, and a straight line into one another in a rectangular combination with components of the same amplitude in X and Y, with equal periods, and with initial phases differing 90° for a circle, 0° for a straight line, and anything but 90° and 0° for an ellipse. The axes of the ellipse are given in 223.

942. *Ellipse with its Center moving in a Straight Line.* This is shown in Fig. 942A, Frontispiece, right side. By changing the initial phases to another quadrant, the ellipse will slant the other way, 942B, Frontispiece, left side. For the equations see 254.

943. *Ellipse with its Center moving in a Circle.* The figure, not here reproduced, is very much like 923, except that the straight line is replaced by a narrow ellipse. The disk made one turn while the pen traced two complex cycles, in which it drew two equal ellipses in opposite directions with unequal speeds. The starting phases were 30° and 90° for one ellipse, and 210° and 90° for the other.

Ellipse with its Center moving in another Ellipse. This figure is readily drawn by two pairs of circular components (223) with unequal

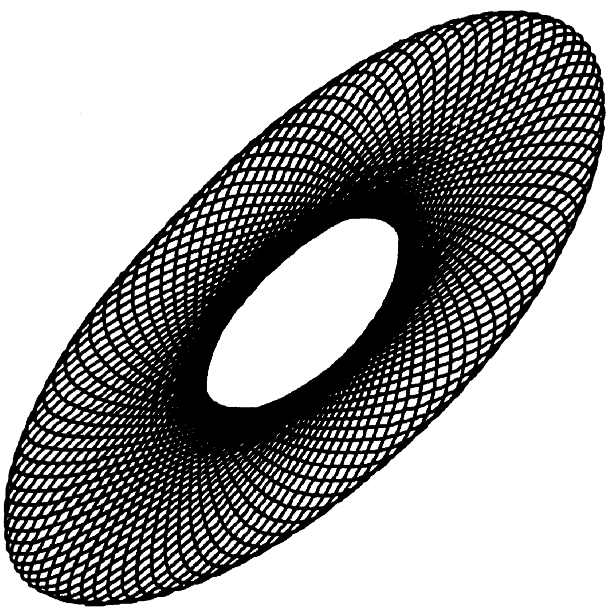


FIG. 943. A Rotating Variable Ellipse.

radii, unequal periods, and unequal slants (942). Thus in Fig. 943 the equation is

$$x = \sin(\theta + 135^\circ) + \tfrac{1}{2}\sin(\tfrac{45}{44}\theta + 45^\circ)$$
$$y = \cos\theta + \tfrac{1}{2}\cos\tfrac{45}{44}\theta.$$

The pen then seems to draw an ellipse clockwise 44½ times, while its axes swing clockwise through 180°. The semi-axes of the ellipse vary from $a = 2$, $b = 0.25$, to $a = 0.8$, $b = 0.25$, to $a = 0.8$, $b = 0.6$, and back again.

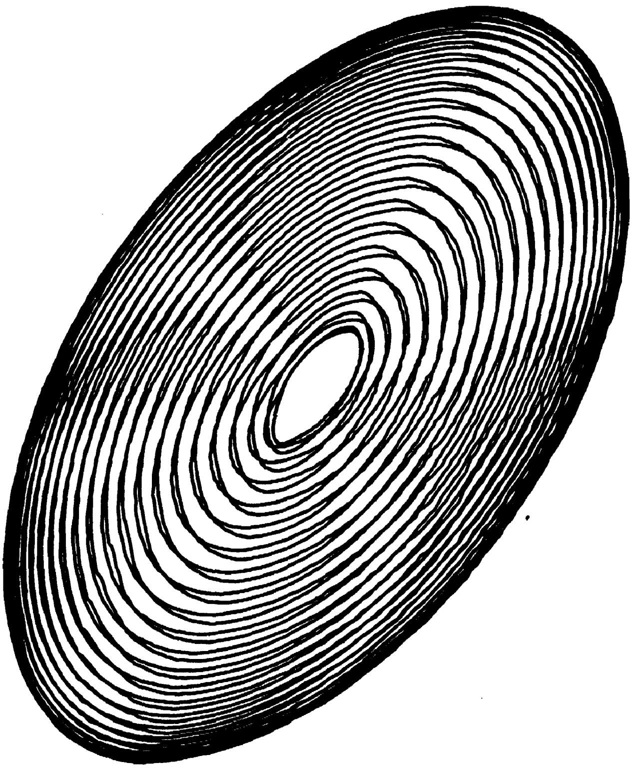


FIG. 944. A Stationary Variable Ellipse.

944. *A Variable Ellipse.* This is an ellipse whose axes vary, its center being fixed. It is drawn like the variable circle in 934 by merely changing ξ, so that, as in Fig. 944,

$$x = \sin(\theta + 30^\circ) + \sin(\tfrac{45}{44}\theta + 30^\circ)$$
$$y = \cos\theta + \cos\tfrac{45}{44}\theta.$$

When the amplitudes are unequal, there is a vacant ellipse in the middle.

945. *A Variable Ellipse with its Center moving in a Straight Line.* This differs from 935 only in the facts that ellipses replace the circles, that they slant, and that the envelope is not symmetrical. The curve was drawn, but is not reproduced.

A Variable Ellipse with its Center moving in a Circle differs from 936 in the same particulars that were just enumerated. It also was drawn, but is not reproduced. Its equation is the same as in 239 except that the starting phases 30° must be inserted in the x terms.

946. *An Ellipse with Fixed Center and Swinging Axes.* When the pen is made to draw two unequal circles in opposite directions and with

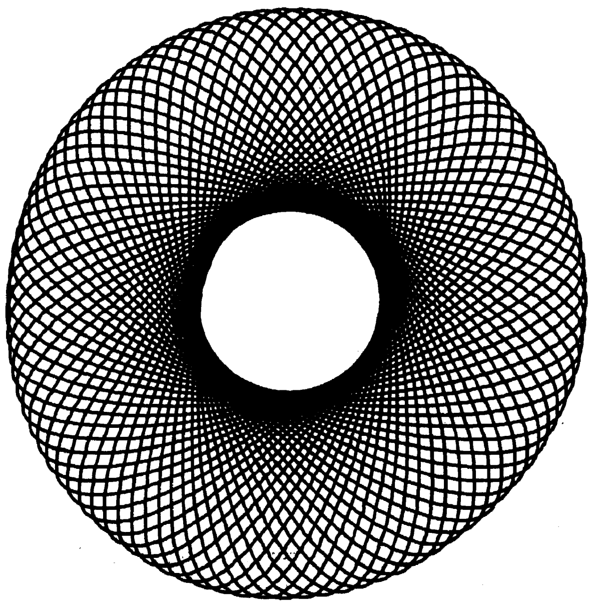


FIG. 946. A Rotating Ellipse.

unequal speeds, as in Fig. 946, it appears to draw the same ellipse with semi-axes $a = 1 + \frac{1}{2}$, $b = 1 - \frac{1}{2}$, which swing round in the same direction as the pen. The ellipse was traced $44\frac{1}{2}$ times while the axes turned 180°.

$$x = \cos 45\theta - \tfrac{1}{2}\cos 44\theta$$
$$y = \sin 45\theta + \tfrac{1}{2}\sin 44\theta.$$

947. *A Rotating Ellipse with its Center moving in a Straight Line.* This Fig. 947 is the preceding one drawn on a moving paper ribbon. In the equation 10θ is added to x.

A Rotating Ellipse with its Center moving in a Circle is easily drawn when the pen traces an ellipse on a rotating disk. It is a simple-looking curve, hardly worth showing.

A Rotating Variable Ellipse with its Center moving in a Straight Line, Fig. 948, is somewhat like 947. 10θ is added to x in 943.

A Rotating Variable Ellipse with its Center moving in a Circle was not drawn, but may also be readily imagined.

A Rotating Variable Ellipse with Fixed Center is given in 943.

949. *A Circle and an Ellipse turning in opposite Directions.* When the pen is made to draw a circle and an ellipse in opposite directions, so that in 924A the phase of one component is changed from 180° to 135° and

$$x = \cos 44\theta + \cos 45\theta$$
$$y = \sin 44\theta - \sin (45\theta + 135°)$$

the pen seems to draw a straight line which widens out into an ellipse of a certain width, and contracts again, twice in the cycle. Half the time the pen rotates clockwise, and half the reverse. The figure 949, Frontispiece, left upper corner, will speak for itself.

V. Rectangular Curve

951. When an ordinary Lissajou curve (223)

$$x = a \sin (m\theta + \xi)$$
$$y = b \sin (n\theta + \eta)$$

is drawn by taking $a = b$, $\xi = \eta = 90°$, and m and n unequal, the pen starts in the upper right hand corner of a square, Fig. 951, and begins its course by drawing a straight line, the diagonal, which is straighter the closer the ratio $m:n$ is. This line, however, is mathematically not really straight, although it may appear to be so to the eye, and at once begins to change into a narrow ellipse. This ellipse widens out more and more, that is, diminishes in eccentricity, until it momentarily, at least, becomes a circle, only to go through the same cycle again in the reverse order, and end in a straight line, the other diagonal. Beginners may thus have a graphic proof, or at least a surmise, that the straight line and the circle are true ellipses at opposite ends of the series. Every ellipse of this series is tangent to the circumscribed square, which math-

ematicians call its envelope. Of course, because the transition is continuous, there is in reality never a true ellipse, nor a straight line, nor a circle, since not one of those drawn is a closed curve. But the words may be used, and the demands of mathematicians satisfied by understanding the word "instantaneous."

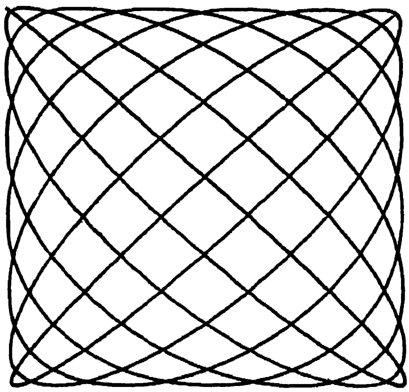


FIG. 951. Rectangular Curve 15 · 14. $\xi = \eta = 90°$

952. When the pen has traced the second diagonal, it has however drawn only half the curve, because as all harmonic curves except the sine varieties* are closed curves, it has not returned to its starting point. If the figure has been well drawn, it may be presented as it is (951), because it has a pleasing appearance. If now the pen is allowed to proceed, it will retrace its entire path in the opposite direction back to the starting point. It will do this with surprising accuracy if the machine is a good one and the starting phases have been well set, so that two apparently complete figures can be drawn in one compound cycle of the pen.

953. The major axes of the ellipses all lie in the first diagonal until the circle stage is reached. Then they suddenly swing over to the second diagonal and remain there until the pen draws the circle the second time, when they return at once to the first diagonal. The major axis, or rather the transverse axis, as mathematicians would prefer to call it, begins with its maximum length, the entire first diagonal, while the minor, or rather the conjugate, axis begins with the value zero. The transverse axis then diminishes to zero and grows again to its maximum

* And also all curves in which the ratio of the periods is incommensurable. This however is not possible when cog wheels are used.

length, while it remains all the time in the first diagonal. The conjugate axis grows from zero to its greatest length and back again to zero, and remains in the second diagonal. At the two moments when both axes are equal (to the side of the square), the ellipse is a circle.

954. The direction in which the pen swings is direct or anticlockwise when X has the faster component and thus leads in phase (223). It is inverse or clockwise during the second half of the time, because when the pen is in the second diagonal, the components are at half phase, 180° apart, and as after that X is more than 180° ahead, it is said to be less than 180° behind, and therefore to lag in phase.

955. When the initial phases are not both $+90°$, the pen does not start from the exact corner of the square, and will not retrace its path.

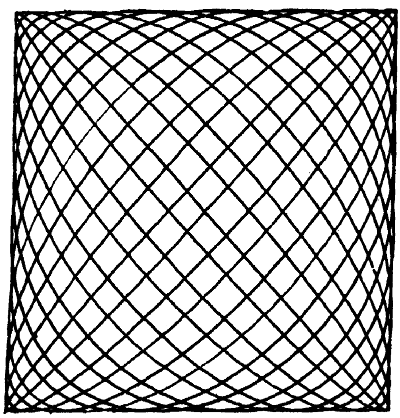


FIG. 955. Rectangular Curve 15 · 14. $\xi = 90°$, $\eta = 78°$.

The two parts of the curve do not overlap, and each half looks defective or unsymmetrical. The lines are unevenly displaced. (Fig. 743C). But they may be displaced very equably by changing one of the starting phases a certain amount. For example, when the periods are 15 : 14 as in 951, the slower component will, after the first round, be $\frac{1}{15}$ of a turn ($= 24°$) behind its companion. Its starting phase should then be made $\frac{1}{2}$ of $\frac{1}{15}$ of a turn ($= 12°$) more or less than 90° ($= 78°$ or 102°). The figure will then have twice as many lines as the first, 955, and its periods will seem to have half the former ratio. (713 A, E, 717 A, E.)

956. When the starting phases are both 90° and the ratio is a close one, as before, but the amplitudes are unequal, the envelope changes from a square into a rectangle. When X has the faster component, the pen swings anticlockwise and clockwise as before, but the axes of the ellipses now turn continually in the pen's direction. The transverse axis varies in length from the diagonal to the altitude of the rectangle, and the conjugate one from zero to the base.

957. In general, the inequality of the amplitudes a and b does no more, of course, than elongate the figure in one direction or compress it in the other. Inequality of the periods m and n makes the pen swing $m - n$ times in each direction, clockwise and anticlockwise. The reason is that, supposing X to be the faster component, the direction is anticlockwise when the phase difference has passed 0°, because then X leads, whereas it is clockwise when the phase difference has passed 180°, for then X lags until it has caught up with its companion in phase. Thus, when the ratio is 16: 15 (in 951, 955), 15: 14, 3: 2 (713), 2: 3 (717), or 2: 1 (743B), the pen swings in each direction only once, whereas in the ratio 15: 22 it swings $22 - 15 = 7$ times.

958. When the speed difference $m - n$ is greater than 1, the pen never traces even an instantaneous ellipse. But when it is 1, and m and n are large numbers, the ellipse is approximated, because for a short time m and n may be considered to be equal, and then the curve is an ellipse (223).

Inequality of initial phases has very litle influence on the nature of a curve when m and n are large numbers, because the components are certain to pass through every phase difference once (or twice if the algebraic sign is ignored) in every compound cycle. Fig. 958 shows such a rectangular curve when $a = b$, $\xi = \eta = 90°$, $m = 44$, $n = 45$, so that the equation is

$$x = \cos 44\theta$$
$$y = \cos 45\theta.$$

See also 743C.

Illustrations in which a simple rectangular (or Lissajou) curve is drawn on a rotating disk are shown in 314, 315, 751, etc., and compound rectangular curves with more than one component in one or both axes in 753.

959. Drawing such a rectangular curve on a moving paper ribbon develops an unexpected feature, in that the envelope is a perfect and

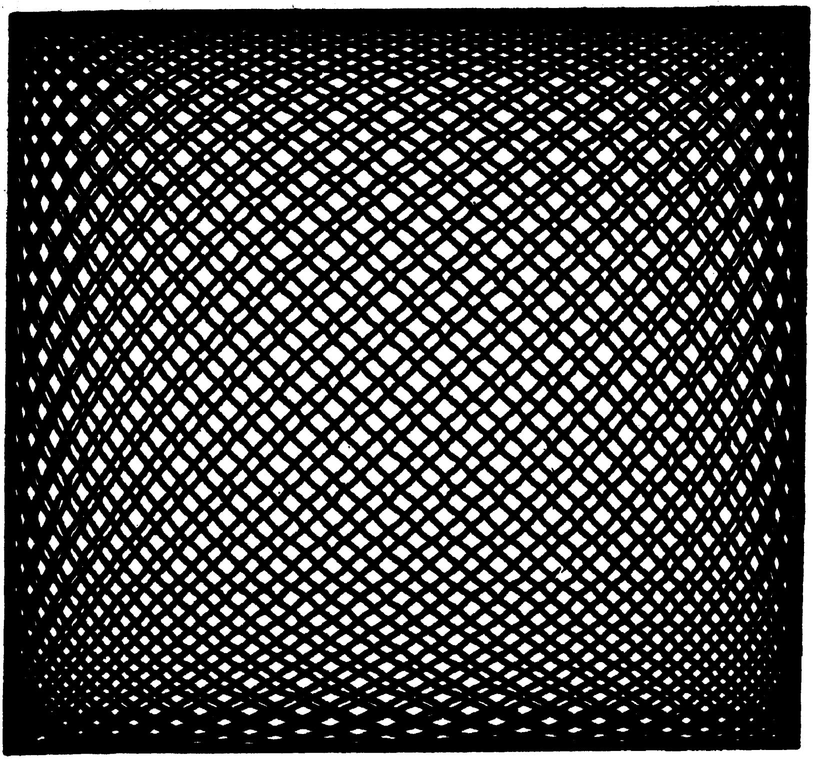


FIG. 958. Rectangular Curve 45 · 44.

well-marked sinusoid, as in Fig. 959A, in the top part of the Frontispiece. The axis of the variable ellipse oscillated between 45° and 135°. When above the envelope, the pen traced the ellipses anticlockwise, and when below it clockwise.

As a rectangular-polar the curve is disappointing, unless the radius could be made very large, and many compound cycles of the pen put into one turn of the disk.

In 959B, the bottom curve of the Frontispiece, a cycloid replaces the sinusoid in A. It was drawn by giving X two components with periods 44 and 45 and starting phases 0°, and Y one component 1.6 as large, with period 45 and initial phase 90°.

VI. Transition Curves

961. When the initial phase of one of the components of a curve is changed successively by the same fraction of 360°, so that its original value is again obtained, there results a cyclic series of figures, such that,

while the transition of each into its next neighbor is sufficiently apparent, distant ones are generally so different that no family likeness whatever can be detected in them. A short series of this kind is given in Fig. 713 and in the article on Cuspidal Rosettes in the Appendix.

But when the initial phases of not only one, but of two, components are changed in turn in this cyclic manner, the complexity and the unexpected variety of the figures is increased enormously. If the draughtsman will not draw these figures in regular order, as he will most probably not do on his own initiative, but will first pick out those which he imagines to have the most prominent initial phases, he will find himself frequently at his wits' ends in guessing at the shapes of the intermediate curves, and then take almost feverish delight in eliciting them from his machine.

962. Such a series of transition curves is given in Fig. 962. They belong to the class of Rectangular Curves. They have two components in each axis, all equal in amplitude, and the periods are such, that one component in each axis has 44 cycles and the other 45. The general equation is

$$x = \sin(44\theta + \xi) + \sin(45\theta + \mu)$$
$$y = \cos 44\theta + \cos 45\theta.$$

The starting phases of the Y components were in every case 90°, while those of X were varied systematically. For the upper row of figures, ξ was placed equal to 0°, and μ taken equal to 0° for the first figure to the left, 30° for the one to the right of it, 60° for the next, and so on up to 360°. In the second row ξ was made equal to 30°, and μ varied as before from 0° to 360° in 30° intervals. For the third row $\xi =$ 60°, and for the fourth $\xi = 90°$, while μ was changed as usual.

963. In the first figure ($\xi = \mu = 0°$), which is the same as Fig. 934, the pen is made to draw two equal circles in the same direction with slightly unequal speeds, the center of one circle being on the circumference of the other. The planes of revolution of the circles are obviously coincident with or parallel to that of the paper.

In the second figure ($\xi = 0°$, $\mu = 30°$), the plane of the second circle may be conceived to be inclined 30° to the paper, so that its projection on the paper is an ellipse, while the plane of the first circle remains parallel to the paper throughout the whole upper row (where $\xi = 0°$). In the third figure the plane of the second circle is inclined 60° to the paper, and in the fourth it is inclined 90° to it, so that it is at right angles, and the projection of the second circle is a straight line.

In the fifth circle ($\mu = 120°$), the inclination of the plane of the second circle is such that, while it is practically equal to 60°, the motion of the pen in it is retrograde.

In the seventh figure (where $\mu = 180°$) the planes of both circles are parallel to the paper, but the motion of the pen is reversed in the second one. This is the same as Fig. 924A, except that here the amplitudes are all equal. (See also 644.)

After this, as μ increases from 180° to 360°, the figures in the upper row are the same as before but in a reverse order and with a negative slant.

In the second row the plane of the first circle is permanently inclined 30° ($= \xi$) to the paper, while that of the second one is made to vary in 30° intervals as usual. This series is not symmetrical like the first one, nor is the third where $\xi = 60°$.

When $\xi = 90°$, in the fourth row, so that the plane of the first circle is at right angles to the paper, the series is again symmetrical like the first row.

964. *The Complete Series.* From the 48 figures here presented in Fig. 962 the general law of transformation may readily be gathered with a little study. This last will be facilitated, if the direct and inverse (in this case clockwise and anticlockwise) rotations are designated by plus and minus signs respectively, the inclinations of 0° and 180° as Circles, *C*, those of 90° and 270° as Lines, *L*, and the rest as Ellipses, *E*. Fig. 964 gives an outline of the complete series for all values of ξ and μ in 30° intervals, the elliptical combinations however having been omitted, as too complicated to sketch. With this completed series it is easy to follow and to understand the successive transitions. But it is full of difficulties and surprises. Let the reader confine himself to Fig. 962, and before he is perfectly familiar with it, cover up a few adjacent figures and then try to supply them in imagination. Or let him do this to a mathematical friend. He will then realize the difficulty and the fascination of the problem.

965. A closer study of Figs. 962 and 964 shows, that if the direction of the slant is not taken into account, there are in reality only 24 different figures* in the complete series of 144, so that there was no need of drawing all of them. In Fig. 964 the variable circle 0° — 0° appears

* These 24 or more count only as one in the enumeration given in 476, for the reason that their periods are all the same. Indeed, they are not counted at all, because duplicate wheels are used in the components.

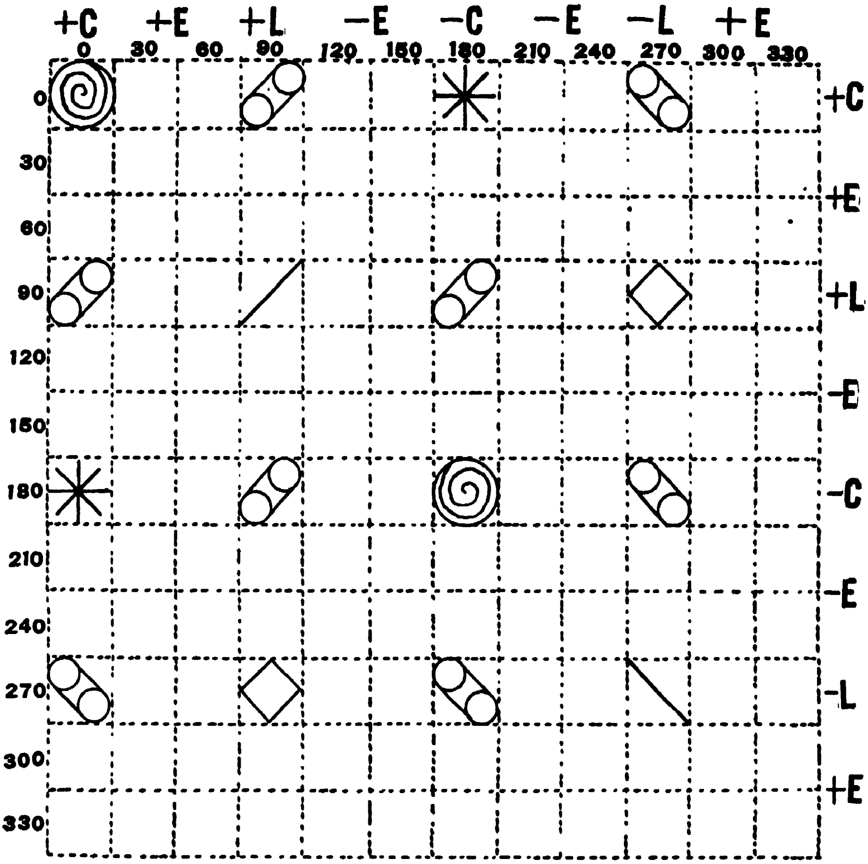


FIG. 964. Scheme of the Complete Series of a Set of Transition Curves.

again in 180° — 180°. The last vertical column, and the bottom row, in which the phase of one component is 360°, are obviously the same as those in which the phase is 0°. Although on the one hand it might have seemed desirable to represent them, on the other hand it was thought better not to do so, for the reason that their presence would give a false idea of the frequency of certain figures in the series. Thus, for example, the variable circle 0° — 0° appears in reality only once, whereas otherwise it might seem to occur four times, once at each corner. But if the missing rows of 360° are supplied in imagination, another feature may be detected in the perfect symmetry of each fourth part of the series, the 180° lines serving as connecting strips. This perfect symmetry is to be found nowhere else.

966. *Adaptation of Transition Curves to the Stereoscope.* Everybody who is at all conversant with Chapter VII of this book, will at once judge the transition curves of Fig. 962 to be an excellent series for the stereoscope. This judgment is however doomed to disappointment. First, the angular interval of 30° between the figures is by far too large. It ought to be only about 5°, but that would demand six times as many figures, a rather discouragingly large number when it takes twenty minutes to draw each one of them. Secondly, as the straight lines 90° — 90° and 270° — 270° belong to the series, they prove that a stereoscopic effect cannot be obtained. For the plane generatrices of a stereoscopic solid are in reality orthographic projections of that solid, and it is evident that the straight line cannot possibly be the projection of a solid. Thirdly, when a stereoscopic solid is turned through the parallactic angle, all its parts are turned equally about the Y axis. Now, in the transition curves of Fig. 962 only one component at a time had its initial phase changed, while that of the other remained the same. Two adjoining figures cannot therefore be different projections of the same solid. They could, of course, have been adapted to the stereoscope, had both the X components had their phases changed proportionally to their periods. This was not done in Fig. 962 because it would have destroyed the simplicity of the transition, and moreover would have made the series what might be called a linear function of only one variable, whereas now it is a function of two independent ones.

967. *Other Transition Curves.* In the transition curves shown in Fig. 962 the initial phases were changed in a cyclic manner. Other curves of this kind may be drawn by varying the amplitudes. In polar curves a mere shifting of the starting position of the pen is sufficient, although initial phases and amplitudes may also be altered. It is thus evident that not only are harmonic curves in general of infinite variety and teeming with agreeable surprises, but even this small section of transition curves bids fair to be inexhaustible. Indeed, this last class may even offer the stronger inducement to one who intends to construct a machine, because it can be drawn with simpler apparatus.

Summary of Chapter IX

I. *Point* (911).

II. *Straight Line* of constant length (921) with its midpoint moving (922, 923).

   A Rotating Straight Line with fixed or moving midpoint (924).

   A Variable Straight Line (925, 926).

   A Rotating Variable Line (927-929).

CHAPTER X

APPENDIX *

Concerning a New Method of Tracing Cardioids

A cardioid may under certain conditions be traced by a point when its motion is the resultant of a rectilinear simple harmonic movement and a uniform angular one of the same period. The harmonic motion may be obtained from any plane mechanical contrivance such as a revolving crank and a sliding slotted bar, while the angular one is best furnished by a disk rotating under the pen.

Fig. 1 will illustrate the definition just given as well as the conditions to be mentioned presently. The point A is the center of the disk which rotates in a clockwise direction with uniform angular speed. If the disk did not rotate, the tracing pen would move over the line EG parallel to the Y axis with simple harmonic motion, so that its distance from R, the middle point, would at any moment be the sine of the phase, the amplitude RE or RG being taken as the unit of our scale in this investigation. But when the disk does rotate, the combination of the rectilinear motion of the pen with the rotary one of the disk causes the pen B to trace the cardioid $BCKQZFB$ (R is only by accident on the curve), provided the following conditions are observed.

Conditions to be Observed.—First, the pen may be set down on the disk as at B at any initial phase α of its rectilinear harmonic motion. In Fig. 1 this initial phase α is taken as 52°, so that $RB = \sin 52°$, RE, as said, being unity.

Second, the point B, at which the pen is set down on the disk, must be on the unit circle whose center O is on the Y axis and whose distance from A, the center of the disk, is the sine of the phase $OA = \sin\alpha = \sin 52° = BR$. The angle AOB we will call β, the *starting angle*, and the circle just mentioned the *starting* circle. In Fig. 1 β is equal to 77°.

Study of the Conditions.—It is to be noted first that α and β are independent variables and may have any values whatever.

Secondly, the initial phase α fixes the center of the starting circle O on the axis of Y, so that O is in the same direction from A that B

* The four articles in this Appendix are reprinted through the courtesy of Prof. Walter B. Ford, Editor-in-Chief of the *American Mathematical Monthly*, who also generously gave the free use of the cuts.

is from R, and $OA = BR$. It also fixes the center D of what we will call the *cusp*-circle, because it is the locus of the cusp of the cardioid that can be generated with the given initial phase α. The radius of this small circle is one half that of the starting circle, or one half of our chosen unit, its circumference passes through A and O, and it is internally tangent to the large circle at S on the axis of X, so that the angle $ASO = \alpha$. This puts S to the left of A when $\cos \alpha$ is positive, and to the right when negative. As α grows from 0° to 360° the center O of the starting circle executes a simple harmonic motion along the axis of Y, about its central position at A, while the center D of the cusp circle moves with uniform angular speed in a circle of its own size about the center of the disk A.

Thirdly, the assumed point B at which the pen is set down on the starting circle, or the assumed value of AOB or β, determines the posi-

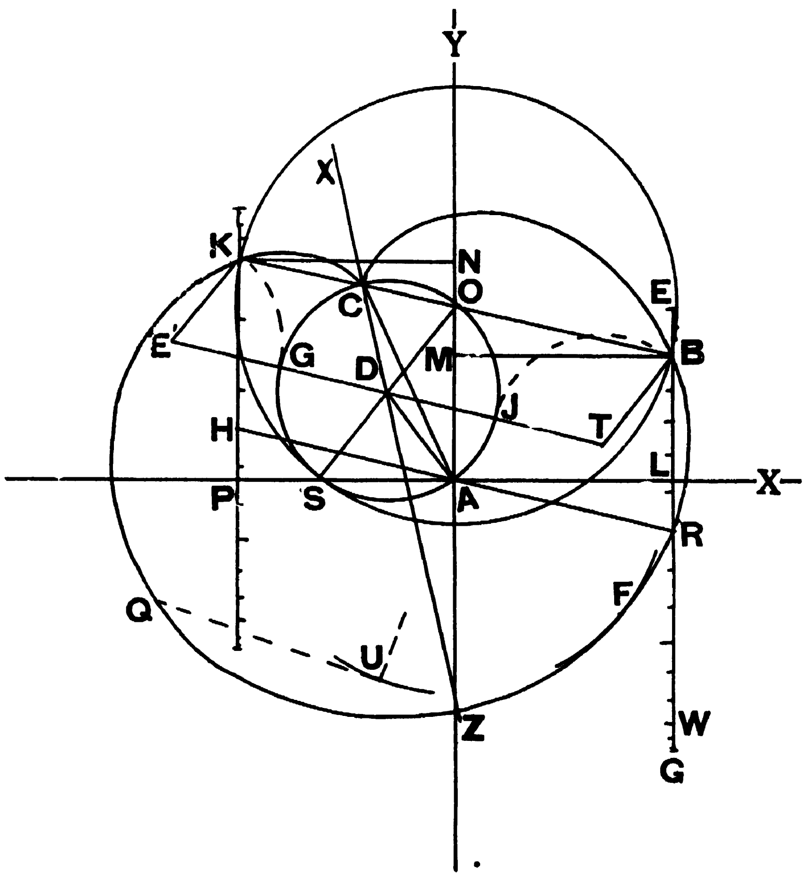


FIG. 1

tion on its circle of the cusp C of the cardioid that will be generated, since this cusp C must lie on the diameter through O and B. The same cardioid is drawn when β is increased or diminished by 180°, and the pen set down at K instead of at B. Measured from their common point of tangency S on their respective circles in a positive or anti-clockwise direction the distances of the cusp C and the starting point B or K are equal, that is, arc $SAJOC = \text{arc } SB$, and the first increased by a whole circumference is equal to the arc SBK, so that as β grows from 0° to 360° the cusp C swings round its circle twice with uniform speed. The axis CZ of the cardioid will of course pass through D.

From what has just been said about α and β, it follows that two or more pens may be set down at various points on the starting circle, fastened together, and made to move with the same simple harmonic mechanism: they will all trace equal cardiods which have their cusps on the same small cusp circle; and a pair of diametrically opposite pens will trace the same cardioid.

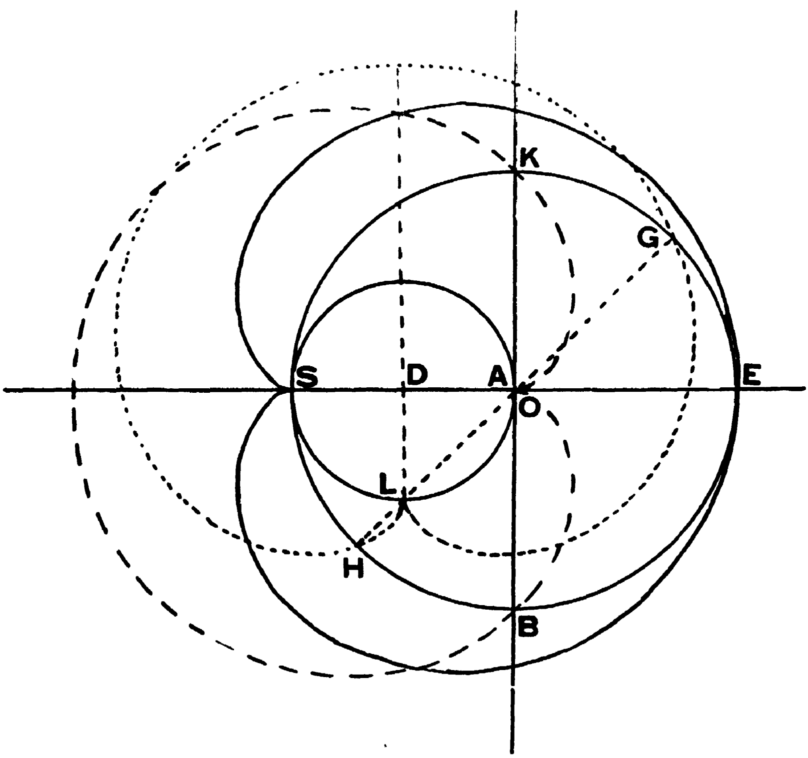


FIG. 2

Other Initial Phases and Starting Angles.—Various figures are here given to illustrate the changes that come upon the position of the cardioid as α and β are given different values, the cardioid however being always of the same size as long as the amplitude of the simple harmonic motion remains the same. We will at this juncture call attention only to Figs. 1, 2, 3. Fig. 1, as said before, gives the values of $\alpha = 52°$ and $\beta = 77°$ or $257°$. In Fig. 2, we have $\alpha = 0°$, when the center O of the starting circle is at A, the center of the disk; the center D of the cusp circle is on the axis of X to the left of A, and the point of tangency of both circles is at S. When $\beta = 0°$ or $180°$, the pen is set down at B or K, and the cusp of the cardioid is at A. When $\beta = 90°$ or $270°$, the pen is started at E or S, and the cusp is at S. The cusp is always at the starting point when this is at the point of tangency or $180°$ away from it. When $\beta = 135°$ or $315°$, the pen is set down at G or H, and the cusp is at L. It will be seen that the cusp is always on the diameter through O and the starting point.

When $0° < \alpha < 90°$ we have a drawing like Fig. 1.

When $\alpha = 90°$ as in Fig. 3, the center O of the starting circle is at the distance unity from A, and the center D of the cusp circle is halfway between them, their point of tangency being at A. When $\beta = 0°$ or $180°$, the pen is set down at A or K. When $\beta = 35°$ or $215°$, the point is B or F and the cusp is at C.

The easiest way to draw a cardioid is to take $\alpha = 90°$ and $\beta = 0°$, that is, to place the pen exactly at the center of the disk A when its phase is $90°$. This is the way in which the writer had drawn a number of cardioids, when it occurred to him to displace the pen from A to some other point such as B, to see what would happen. Fortunately B was on the starting circle, and to his great surprise a perfect cardioid resulted. But the position of B had not been marked, so that in endeavoring to rediscover it later, hours of experimentation were necessary before its position was determined to be on the starting circle. This was changing β. Then the idea came to change the initial phase α. After much more experiment and analysis, the final law was found which is presented in this article.

When $90° < \alpha < 180°$, we have a diagram like Fig. 1 when this is turned half-way round the Y axis and looked at from the other side of the paper, the points D and S then being to the right of OA.

When $\alpha = 180°$, we have Fig. 2 when the paper is turned half-way round the point A with the same side of the page facing us. For the values of α from $180°$ to $360°$, the points O and D are below the X axis, and the figures are obvious enough after a little study.

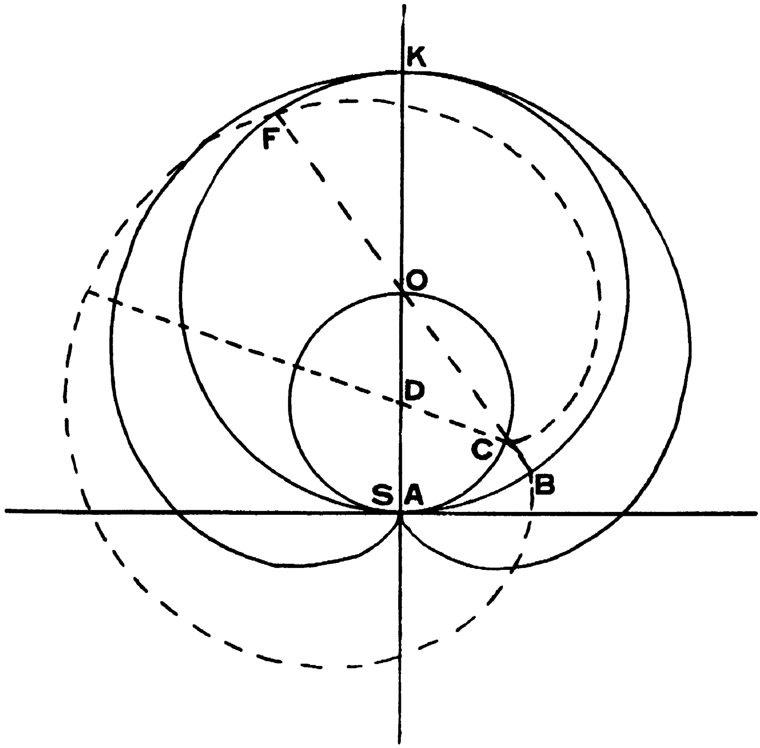


FIG. 3

Object of this Article.—The chief object of this article is to prove that under the given conditions the curve so drawn is a cardioid.

The method to be pursued in the proof will be to suppose the cardioid generated as an epicycloid, and then to show that the pen traces the same curve. Taking the cusp circle, then, as the fixed one, the cusp C must evidently lie upon it and the axis of the cardioid must pass through its center. Placing the equal generating circle with its center at T in Fig. 1, when its tracing point is at B, we have the arcs COJ and BJ equal, and the lines CB and DT parallel. As the quadrilateral $DOBT$ has its opposite sides equal, they are also parallel, and hence OB is likewise parallel to DT, so that the lines CB and OB are coincident, and therefore the cusp C lies on the diameter KOB.

The Phase of the Pen at the Cusp.—For the sake of greater clearness we will suppose the disk to remain at rest and the pen to have its rotary motion in the opposite sense, that is, in an anti-clockwise direction. We will also assume for the present that the pen really traces the epicycloidal cardioid drawn by revolving the generating circle T

with its pen at B about the fixed circle D. This assumption will be given a rigorous proof later, but it will serve well now to establish some of the conditions, such as the position of the cusp C and the phase of the pen when at the cusp.

It is evident that in passing through the cusp C the pen must be momentarily at rest on the paper and its rectilinear and rotary velocities must be equal and opposite. The diagram offers the suggestion that the pen is then at one of its two least distances from the center of revolution A, and that AC must be equal to $AL = MB = \sin\beta$. As there are two points on the cusp circle at this distance from A, we must take the one that is on the diameter KOB, as said before.

We must next find the two phases of the pen in its rectilinear motion when it is at L, at its minimum distance from A. As R is its mean position at the phase 0°, LR is equal to the sine of the phase when at L. Now as AO is equal and parallel to BR, OB and AR must also be equal to one another and parallel so that the triangles BOM and ARL are equal, and then $LR = OM = \cos\beta = \sin(90 \pm \beta)$. The phase of the pen when at the cusp is then either $90° - \beta = 90° - 77° = 13°$, or $90° + \beta = 90° + 77° = 167°$. In this instance we must take the latter value, because the direction of the linear motion of the pen must be downward at L in order to be to the right at C and annul its rotary motion which is counter-clockwise or to the left at C.

The rectilinear speed of the pen at L, or at C, is then $d[\sin(90° + \beta)] = \cos(90° + \beta)d\beta = -\sin\beta d\beta$. The rotary velocity at C is $CAd\theta = +\sin\beta d\theta = +\sin\beta d\beta$, because $d\theta$, the angular velocity of the pen, $d\beta$ or $d\alpha$, the variation of the rectilinear phase of the pen, are all equal and constant. Hence as the rectilinear velocity of the pen is $-\sin\beta d\beta$ and its rotary speed is $+\sin\beta d\beta$, it is at rest at the cusp C and is then in the phase $90° + \beta$. The pen is in phase $90° - \beta = 13°$, in the present instance, when at F, where the rectilinear and rotary motions are also equal but in the same direction. There can be no other points besides C and F at which the two motions of the pen are exactly equal and either in directly opposite or in the same direction, because as the rotary motion must be at right angles to the radius through A, the two L points of the rectilinear motion are the only ones in which this is true.

As CA in the cusp circle is equal to $\sin\beta$, it follows that the angle $CDA = 2\beta$, and $DCA = DAC = 90° - \beta$. The angle $DSA = \alpha$, since $OA = \sin\alpha$ and $SO =$ unity, and hence also $SA = \cos\alpha$, and $DAS = OCA = \alpha$. The position of the axis of the cardioid is therefore readily found. It is only by accident however that the cardioid passes through R, and that Z seems to lie on the Y axis.

The Second Starting Point K.—On account of the equality of the angles SDJ, SOB, DTB, CDJ, the (reflex) angle CDS is double the angle SOB, so that if the angular positions of the pen at B and of the cusp C are reckoned from the common point of tangency S of their circles, the angle of the cusp is twice that of the initial position of the pen and the arcs $SAJOC$ and SB are equal. If we add equals to these equals, the whole circumference of the D circle to the first and a semi-circumference of the O circle to the second, it follows that the pen may be set down at K as well as at B in order to trace the same cardioid. The explanation given before for the point B then applies equally to the point K, if we replace LR by HP, BR by HK, and remember that as β is then increased by 180° or $KOA = 180° - \beta$, NO and HP are equal to $-\cos\beta = -\sin(90° \pm \beta) = \sin(270° \pm \beta) = 347°$ or 193° in the present case. The first of these $270° + \beta = 347°$ must evidently be here the phase of the K pen when at the cusp C, because the rectilinear motion at P must be upward, or to the right at C, to counteract the left rotary motion. As $90° + \beta$ and $270° + \beta$ differ 180°, the B and K pens are half a phase apart. And as their linear distance from one another is $KOB = 2 = (1 - \cos\theta) + (1 + \cos\theta)$, the line joining them passes through the cusp.

Preliminaries to the Proof that the Pen Traces the Cardioid.—We are now in a position to verify the assumption that the pen B traces the epicycloidal cardioid. First let us examine the rectilinear component of its motion. This would at any phase interval θ after passing through L, bring it, say, to W, so that its distance from L would be $LW = RW + RL = \sin[(90 + \beta) + \theta] - \cos\beta$. Here $90° + \beta$ is the phase of the pen at L, as we saw before, so that $(90° + \beta) + \theta$ is the phase after the interval θ. Taking $\theta = 135°$ as an example, this would make the phase $167° + 135° = 302°$, and put the pen at W, RW being equal to $\sin(90 + \beta + \theta)$. LW is in principle equal to the difference of RW and LR, that is, $RW - LR$ as would be evident if W were between L and R and both RW and LR positive. Here however, as RW is minus and LR plus, we have their numerical sum, which is minus. As $\sin((90° + \beta) + \theta) = \sin(90° + (\beta + \theta)) = \cos(\beta + \theta)$, the length of $LW = \cos(\beta + \theta) - \cos\beta$.

Secondly, the rotary component alone of the pen after its arrival at the cusp in the phase $90° + \beta$, during the phase interval θ after that, swings the pen from C to U through the angle $\theta = CAU$, which, as said, is here taken as 135°, the center of rotation being A, and the radius $UA = CA = AL = \sin\beta$.

Thirdly, in compounding the two motions of the pen, if we allow the rotary motion first to carry the pen from the cusp C to the point U,

and then the rectilinear component to move it from U to Q over the tangent $UQ = LW = \cos(\beta + \theta) - \cos\beta$, we must, if our contention is correct, find the point Q on our cardioid with a position angle $QCX' = \theta$. The actual proof however will be the reversal of this procedure, taking a point Q on the cardioid at the position angle $X'CQ = \theta$, and showing that the length of the tangent QU dropped from it to the A circle is equal to $\cos(\beta + \theta) - \cos\beta$.

The Proof.—Taking C as the pole or origin and CX' as the positive direction of the X axis, the equation of the cardioid is $\varrho = 1 - \cos\theta$, $SO = OB$ [$= \frac{1}{2}CZ$] being unity.

The coördinates of Q are then
$h = \varrho\cos\theta = \cos\theta - \cos^2\theta$;
$k = \varrho\sin\theta = \sin\theta - \sin\theta\cos\theta$.

The radius of the A circle is $AU = AC = r = \sin\beta$.

The coördinates of its center are
$a = -AC\cos DCA = -\sin\beta\cos(90° - \beta) = -\sin^2\beta$;
$b = -AC\sin DCA = -\sin\beta\sin(90° - \beta) = -\sin\beta\cos\beta$.

The square of the tangent QU dropped from a point (h, k) to the circle $(x - a)^2 + (y - b)^2 - r^2 = 0$ being $(h - a)^2 + (k - b)^2 - r^2$, we here have the square of QU equal to $(\cos\theta - \cos^2\theta + \sin^2\beta)^2 + (\sin\theta - \sin\theta\cos\theta + \sin\beta\cos\beta)^2 - \sin^2\beta$.

Upon reducing, we find this value equal to the square of $\cos(\beta + \theta) - \cos\beta$. Therefore the pen B traces the cardioid $BCKQZFB$.

We may notice that the initial phase α does not appear in this analysis, because that merely fixes the positions of the centers of the starting and cusp circles O and D on the disk. The point B, however, with its starting angle β, being a point on the cardioid, determined the actual position of the cusp and the direction of the axis CZ. And lastly although definite numerical values of α, β, and θ, have been used in the figures for purposes of illustration, only general algebraic values have entered our equations.

Analysis and Synthesis of the Motions of the Pen.—The analysis just presented shows us how the cardioid is drawn mechanically. While the rotary motion swings the pen around the *tangent* circle (as we may call it) with uniform speed (see also Fig. 4) the rectilinear harmonic motion keeps it on the tangents to the successive points at the distance $\cos(\beta + \theta) - \cos\beta$. If we wished to plot these tangents as in Fig. 4, we would have to begin with a knowledge of the zero direction of the rotary motion. Its center is, of course, at A, and we know that the cusp C must be in the phase $90° + \beta$ ($= 167°$). Centering a protractor at A so that C reads $90° + \beta$, as in Fig. 4 in which the radii are drawn in

dotted lines at intervals of 30°, we shall not only easily find the zero, but also see that L will indicate the initial phase α ($=52°$), F will show $90° - \beta$ ($=13°$), and D 180°.

As we wish to plot our tangents from the true 0° of the rotary motion, instead of from the cusp C from which we before reckoned the phase interval θ in our proof, we must subtract $90° + \beta$ from the θ there used. Changing back our $\cos(\beta + \theta)$ into $\sin(90° + \beta + \theta)$ from which it originated, and then subtracting $90° + \beta$ from θ, we have for the length of a tangent $\sin\theta - \cos\beta$, which, as $\cos\beta$ is constant, gives truly a rectilinear simple harmonic motion.

After the cardioid has actually been drawn, there is a very simple and expeditious way of finding at any time the phase of the pen at any

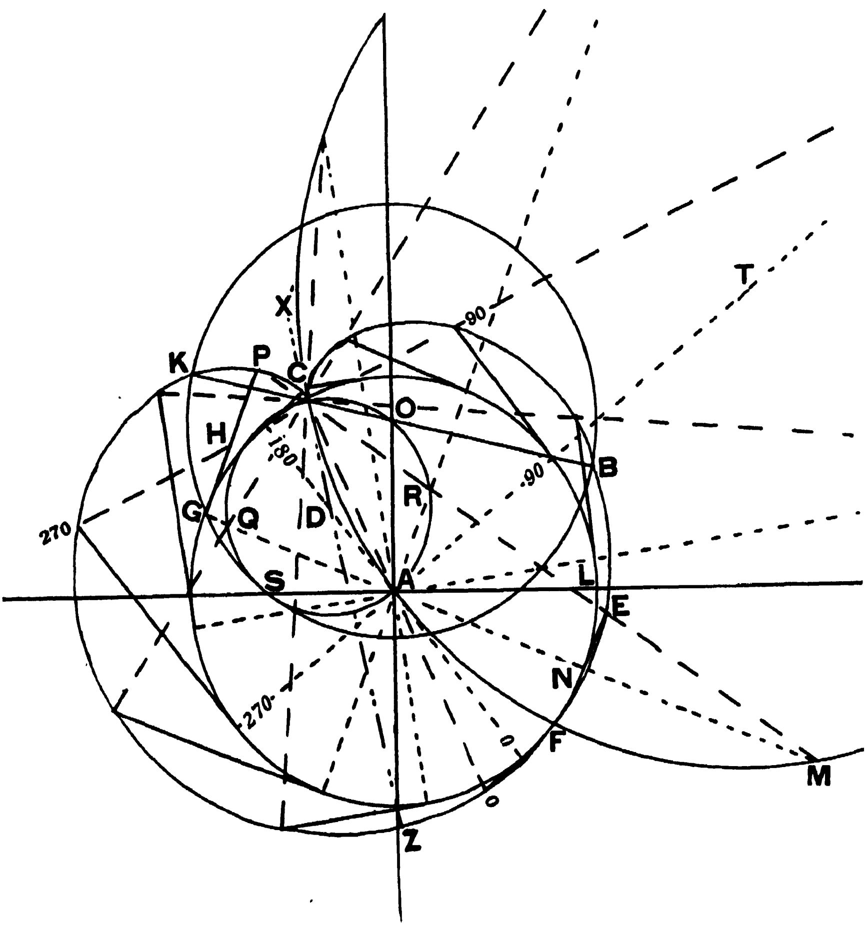


FIG. 4

point or conversely its position at a given phase. We need but center a protractor on the cusp C and take CA as the zero line. This has been done in Fig. 4, in which the radii vectores are drawn in dashed lines. The phases of the pen are then shown directly by the readings on the protractor. B will thus be found to show the initial phase α ($=52°$), K will be $180° + \alpha$ X' $90° + \beta$ and Z $270° + \beta$. The points of the cardioid at the phases 90° and 270° will be seen to have maximum, and those at $90 \pm \beta$ minimum distances from A.

While the mechanical method used in drawing the cardioid is really the tangent method just mentioned, the components of the movements, that is, the uniform angular speed of the pen about the center of the disk A, as well as its rectilinear simple harmonic motion, are bound to disappear individually from view on account of their combination. But if we refer now the motions of the pen to the cusp C, we find that the pen really moves about this cusp with uniform angular speed and also really moves in its direction with true simple harmonic motion, so that while the compound motion of the pen about the center of the disk is such as to make its components indistinguishable, these components, although united, are elegantly re-analyzed and exposed to view when we refer the motion to the cusp.

The Intersection Circle.—A study of Fig. 4 will reveal the peculiar property that the radii vectores passing through the cusp C and the radii of the tangent circle centered at A, that are in the same phase, intersect each other on what we may call the ***intersection*** circle which has its center at T. For example, the radius vector $PCREM$ of 30° (or 210°) and the radius $GQANM$ of the same phase, intersect in M which is on the intersection circle T. That this is really the case is due to the fact that the angle at M is constant and equal to $90° - \beta$. The 0° lines of the radii centered at C and A make this angle $CAD = 90° - \beta$, with one another, as we saw before in Fig. 1, so that all other radii in the same phase must also be inclined to one another at this angle, and hence the angle $PMG = 90° - \beta$. Therefore the intersection points of all the radii in equal phases lie on a circle, because they form triangles like MCA which has a constant base CA and a constant vertical angle M.

The length of the tangent NE (at 30°) is $+\sin\theta - \cos\beta$ and of GP (180° away) is $-\sin\theta - \cos\beta$ or $\sin\theta + \cos\beta$, if regard only its absolute value. Their difference $PH = 2\cos\beta$, and their distance apart $GN = 2\sin\beta$. Let us imagine the line HE drawn parallel to GN which has been omitted in order not to crowd the figure too much. Then we have the proportion

$$\frac{PH}{HE}=\frac{PG}{GA+AM} \quad \text{or} \quad \frac{2\cos\beta}{2\sin\beta}=\frac{\sin\theta+\cos\beta}{\sin\beta+AM}$$

from which we find the chord $AM=\sin\theta\tan\beta$. This chord becomes a maximum when $\theta=90°$, so that the diameter of the intersection circle is $\tan\beta$ and its radius $AT=\frac{1}{2}\tan\beta$. Its center T lies on the $90°$ — $270°$ radius through A, produced if necessary, and its circumference passes through C, A and F.

The points R and Q, at which a radius vector through C and a radius through A differing $90°$ in phase intersect, lie on the cusp circle, because the quadrilateral $CRAQ$ has two opposite angles at C and A equal to $90°$, so that the other two are supplementary and are therefore inscribed in the circle passing through A and C.

In Fig. 5, α has its old value of $52°$, but AOB or β has been made equal to $90°$. The starting and cusp circles are in the same positions

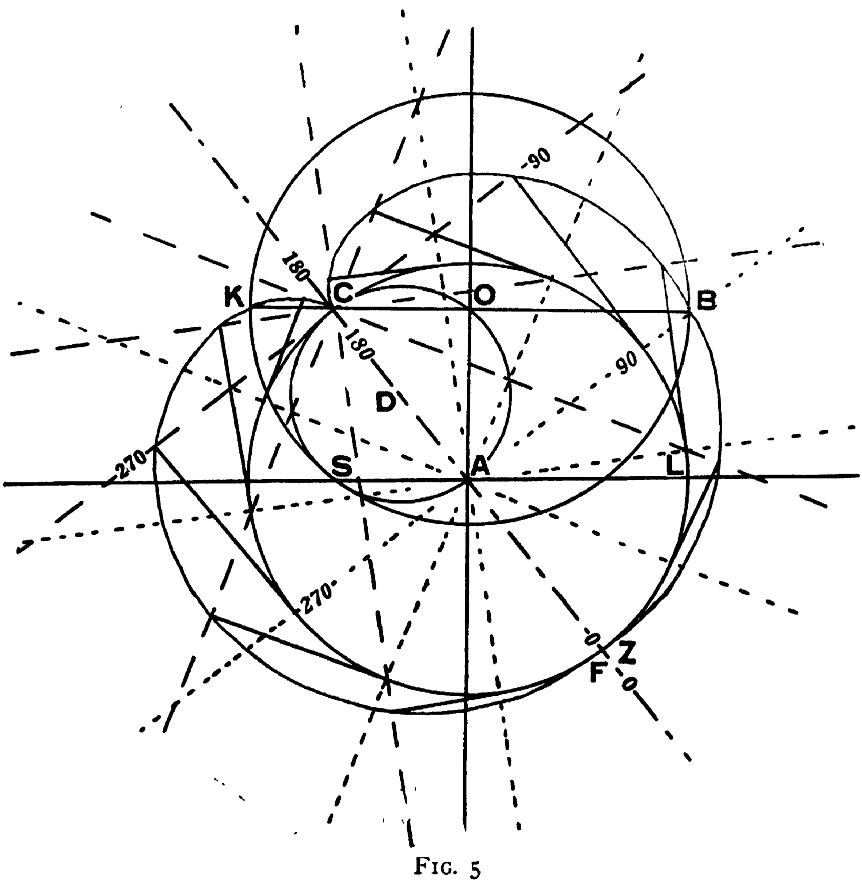


FIG. 5

as in Fig. 1, because these depend upon α alone. The cusp C however, has shifted somewhat, so that it is now on the line DA since the angle $CAD = 90° - \beta = 0°$ in the present case. F is also on the line CDA, which is here the common 0° line of the radii passing through C and A. The radius AC or AL of the tangent circle is unity, and the lengths of the tangents $\sin\theta - \cos\beta$ drawn to it are now simply $\sin\theta$. The radius of the intersection circle $\frac{1}{2}\tan\beta$ is infinite, so that the radii through C and A at equal phases are parallel.

In Fig. 6, α is as usual 52°, but $\beta = 0°$. The starting and cusp circles and the 0° line DA are the same as before, but the cusp C is now at A. The radii of the tangent and intersection circles are zero. The lengths of the tangents $\sin\theta - \cos\beta$ are $\sin\theta - 1$ or $1 - \sin\theta$, and are ordinary radii vectores of the cardioid.

Finally in Fig. 7, $\alpha = \beta = 52°$. The starting and cusp circles and the 0° line of the tangent circle are the same as usual. The cusp C however is now at O, and the 0° line of the radii vectores through it is the negative direction of the Y axis. F has shifted somewhat and the intersection circle is much reduced.

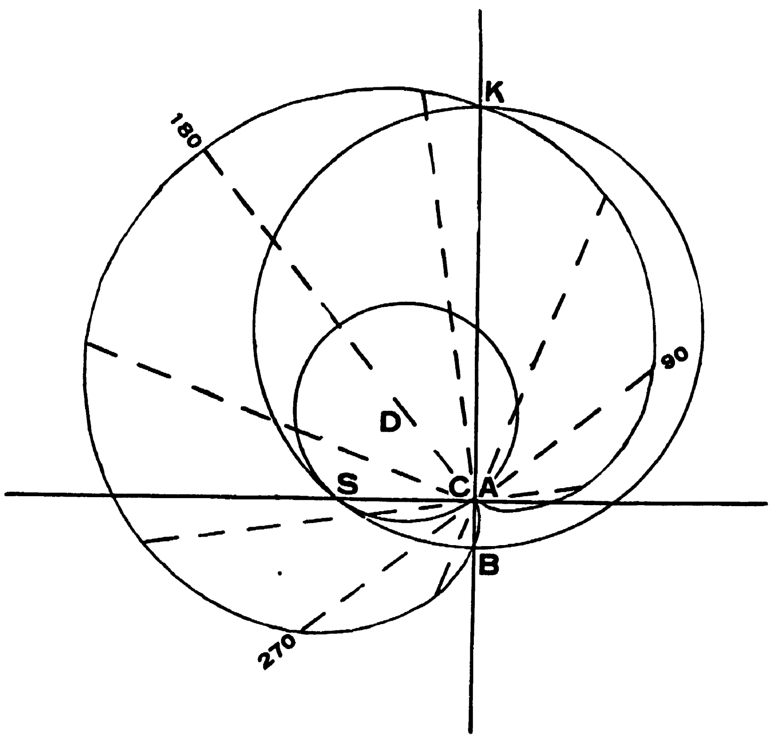


FIG. 6

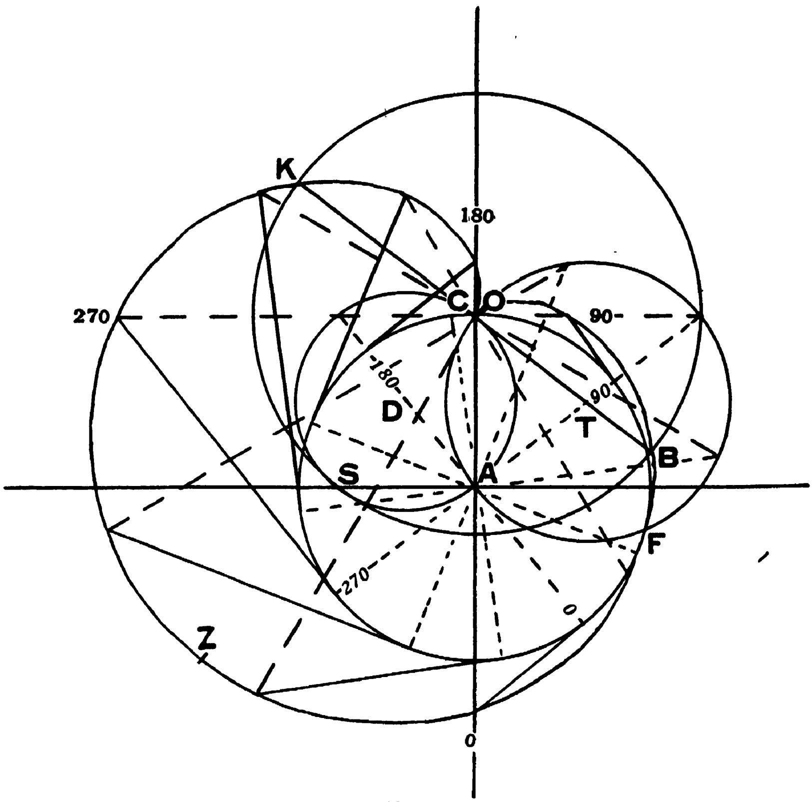


FIG. 7

CUSPIDAL ROSETTES

The rose, rosette, rosace, Rosenkurve or multifolium, or whatever other name it may have, is a periodic polar curve whose equal sectors may have any angular magnitude. Its general equation, as usually given, $\varrho = a + b \sin n\theta$,* supposes the tracing point to move with a simple harmonic motion of n cycles along a radial line through the pole, at the same time that it makes one revolution about this pole with uniform angular speed. A simple instance of such a polar curve is the trifolium, Fig. 1, which is drawn by having a tracing pen move with simple harmonic motion of amplitude b in a radial line over a uniformly rotating disk in such a way that the pen just touches its centre without passing beyond, and the disk makes one revolution in three cycles of the pen. The equation is then $\varrho = a(1 - \sin 3\theta)$, as is seen by inspection in Fig. 2.

* The two terms may have unlike signs and sin be replaced by cos.

If in the general equation a is greater than b, the pen does not reach the center as in Fig. 3, and if a is less than b, the pen passes beyond it as in Fig. 4 and draws smaller lobes or loops on the other side, which are within or between the larger ones according as n is odd or even, until when $a = 0$, the equation becomes $\rho = b \sin 3\theta$, the loops are equal and we have a trifolium in Fig. 5 somewhat like Fig. 1 but half its size and traced twice by the pen because n is odd, the number of the lobes being doubled when n is even. The rosette is *cuspidal* only when $a = b$ as in Figs. 1 and 2, because only then the pen, after tracing one branch of the curve and coming to a momentary standstill, retreats along another branch, so that both branches have a common rectilinear tangent at the point of rest. This tangent is always between the two branches in the curves treated in this discussion, so that the cusps are of the first species.

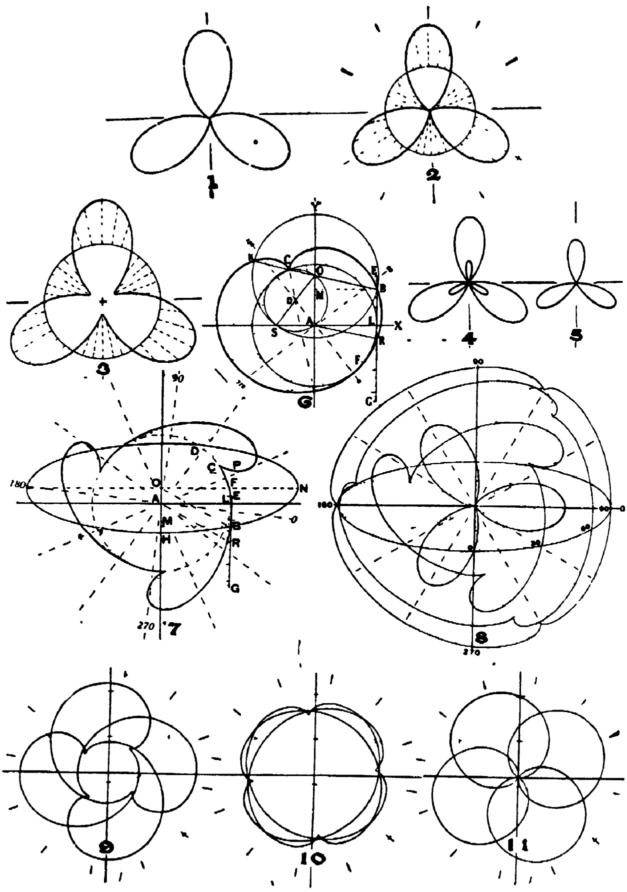
All this is well known and has been mentioned merely as an introduction to the subject in hand. As Moritz* has very completely treated the case of the pen moving along a *radial* line through the center of the disk, the problem now is to investigate what happens when the pen moves along a *non-radial* line, and when it is set down at any initial phase. One who has tried the experiment will know that he obtained a rosette distorted and somewhat like one of the three mentioned before in Figs. 3, 2 and 4, which we may call rounded or curtate, cuspidal, and looped or prolate. Of these the cuspidal rosettes require certain conditions which it is the specific object of the present article to study.

The Cardioid.—As the cardioid may be defined as a cuspidal rosette with the ratio $n = 1$, the present investigations are based upon the foregoing article by the writer, to which the reader must have recourse for more detailed explanations. The following condensed resumé may however be sufficient.

In Fig. 6, A is the center of the disk which rotates with uniform angular speed in a clockwise direction. B is the point at which the tracing pen is set down in any initial phase of its rectilinear simple harmonic motion which has the same period as the disk, so that, if the disk did not rotate, the pen would move over the line ERG, its distance from R, its middle point, being at any moment the sine of the phase.

In order to draw a cardioid it is essential that the point B, at which the pen is set down on the rotating disk, should be on the "starting circle" whose radius OB is taken as unity and is equal to the amplitude of the harmonic motion ER or RG, and whose center O is on the Y axis at the distance from the center of the disk equal to $OA = BR =$

* "On the Construction of Certain Curves Given in Polar Coördinates" by R. E. Moritz, in the May, 1917, *Monthly*.



$\sin \alpha =$ the sine of the initial phase. The cusp C of the cardioid will then be on the cusp circle, with radius one half of OB, which is internally tangent to the starting circle at S on the axis of X so that the angle $OSA = \alpha$, and which passes through O and A. The selection of the point B on the starting circle fixes the starting angle $\beta = AOB$, which then locates the cusp C on the diameter through B. The distance of the cusp C from A is equal to $AC = AL = MB = \sin \beta$. There is also a second starting point K, diametrically opposite B, so that $\beta' = 180° + \beta$, upon which the pen may be set down at the same initial phase α and trace the same cardioid.

The mechanical method mentioned of tracing the cardioid is what we might call the *tangent* method. A similar method will be seen to apply in a more generalized form to cuspidal rosettes of all kinds. To

facilitate the comparison, the same letters and symbols will be used as much as possible.

Cuspidal Rosettes: the Ratio $n = 3$.—We begin with a simple ratio of $n = 3$, or $n = p/q = 3/1$, that is, p or 3 cycles of the rectilinear motion of the pen being equal to q or one of the disk, Fig. 7. It is immaterial whether we suppose both the pen and the disk to move in the manner indicated, or imagine the disk at rest and credit the pen with its motion in the reverse direction. The initial phase α of the rectilinear motion in Fig. 7 has been taken as 20°, so that $BR = OA = \sin 20°$. We will take $\beta = 30°$. In comparing Figs. 6 and 7 we may note generic similarities and obvious differences. The starting point B and the rectilinear path ERG are the same in both figures, although the scale is not. The starting circle with radius unity in Fig. 6 becomes the large ellipse in Fig. 7, whose center O is as before on the Y axis at the distance $\sin\alpha$ from A. The conjugate axis of the ellipse OH is unity, and its transverse axis $ON = n = 3$, and is parallel to the X axis. AL is now $n \sin\beta = 3 \sin\beta$.

The Starting Point B.—That the point B must be on the ellipse mentioned may be proved by substituting the general ratio n for its value of unity in the cardioid. If in Fig. 6 we draw BO parallel to RA, these lines are equal, and $OA = BR = \sin\alpha$. Drawing BM parallel to AL makes them also equal to one another, and OM equal to LR. As LR is the sine of some phase of the harmonic motion, we may take it as the cosine of the angle ARL or its equal MOB, that is, as $\cos\beta$. Then $MB = AL = \sin\beta$ and $OB = AR = OS =$ unity. Therefore the starting point B is on the circle with radius unity and with its center O the distance $\sin\alpha$ from A.

In the case of a rosette as in Fig. 7, we have MB also equal to AL, $OA = BR = \sin\alpha$ and $LR = OM$. Taking these last equal to $\cos\beta$ as before, we cannot now make AL and MB equal to $\sin\beta$, but must take them equal to $n \sin\beta$, because the variation of the phase of the pen along the line ERG is now n times the angular speed of the disk. Hence the starting point B is on the ellipse whose semi-major axis $ON = n$ and semi-minor axis $OH =$ unity, and center O at distance $\sin\alpha$ from A.

The Locus of the Cusps C.—The rosette in Fig. 7 has three cusps, and in general it is evident that the number of cusps must be equal to p, the number of cycles of the pen, and that their angular intervals must be $360°/p$. For this reason we may confine ourselves to the cusp C that is first drawn after the pen has been set down at B and call it *the* cusp.

To find the phase of the pen when it is at the cusp, we observe that the pen must then be momentarily at rest, so that its rectilinear speed must be equal and opposite to its rotary velocity. As the latter is always at right angle to the radius through A, the harmonic motion can be so only when the pen crosses the X axis at L, so that one value of RL is the sine of the phase of the pen when at the cusp C and the other when it is at F, where the linear and rotary speeds are also equal but in the same direction. The rotary speed of the pen when at C is $ACd\theta = ALd\theta = n \sin \beta d\theta = \sin \beta d\beta$, its rectilinear speed at L is $d(LR)$ so that $LR = \int \sin \beta d\beta$ is equal to $\cos \beta$ or $\sin (90° \pm \beta)$ in absolute value. Of these two, $90° + \beta$ must be the phase for the cusp C, because as the rotary motion carries the pen anti-clockwise, the rectilinear motion at L in Fig. 7 must then carry it in the opposite direction, that is, clockwise or downward, so that the phase must be greater than 90°. For the point F the phase must then be $90° - \beta$.

The phase of the harmonic motion at the cusp C being $90° + \beta$, that of the rotary motion must be $(90° + \beta)/n$. The angular position of C on the disk therefore varies directly as β, and as its linear distance from A is $n \sin \beta$, its locus must have an equation like $\rho = n \sin n\theta$, and when $n = 3$ as in Figs. 7 and 8, this is the equifoliated and non-cuspidal trifolium shown in dotted lines in Fig. 8, which is exactly like Fig. 5 but n times as large. The position of the axis of the first lobe is found by making $n \sin \beta$ a maximum, that is, $\beta = 90°$, so that (see Fig. 8 in which four rosettes are given with $\beta = 0°, 30°, 60°, 90°$) the phase of the pen $90° + \beta$ becomes $90° + 90° = 180°$, and the phase of the disk $180/n$. This last value directly locates the 0° phase of the disk, which may however be found also from any value of β, since it is equal to $(90° + \beta)/n$ as said before.

The arc AC of the cusp folium is equal to the elliptic arc HB on Fig. 7, and similarly to any corresponding elliptic arc on Fig. 8, that is to say, the distance of the cusp C from the center of the disk A as measured along the arc of the cusp folium is equal to that of the starting point B from the Y axis as measured along the ellipse. In the rectification of a polar curve the length of an arc is

$$s = \int (d\rho^2 + \rho^2 d\theta^2)^{\frac{1}{2}},$$

so that in the cusp folium where $\rho = n \sin n\theta$, we have

$$s = \int (n^2 \cos^2 n\theta \cdot n^2 d\theta^2 + n^2 \sin^2 n\theta \cdot d\theta^2)^{\frac{1}{2}}$$

$$= n \int_0^{\beta} \left(1 - \frac{n^2 - 1}{n^2} \sin^2 n\theta\right)^{\frac{1}{2}} n d\theta,$$

which is like

$$s = a \int_{\varphi 0}^{\varphi 1} (1 - e^2 \sin^2 \varphi)^{1/2} d\varphi,$$

the length of an arc of an ellipse with semi-major axis a, eccentricity e, and eccentric angle φ, as given by Byerly in his *Integral Calculus*, page 121, because in our ellipse $a = n$, $b = 1$, $e^2 = (a^2 - b^2)/a^2 = (n^2 - 1)/n^2$. Therefore the cusp folium arc AC equals the elliptic arc HB, and the perimeter of one lobe of the cusp folium is equal in length to the semiperimeter of the ellipse. This expression is true for values of $n > 1$, but when $n < 1$ it takes the form

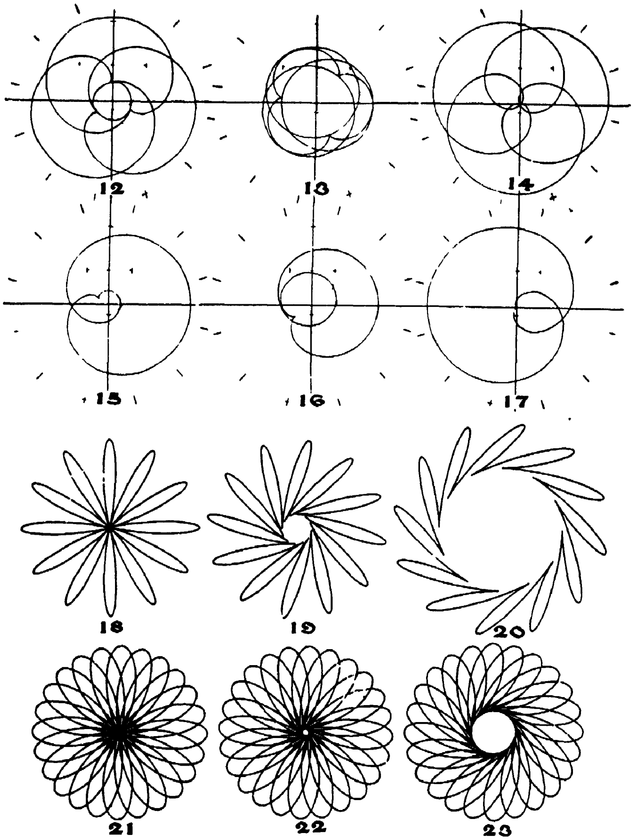
$$s = \int (1 - (1 - n^2) \cos^2 n\theta)^{1/2} n d\theta$$

$$= b \int (1 - e^2 \cos^2 \varphi)^{1/2} d\varphi,$$

so that as a and b then exchange names and a cosine appears in the place of a sine, the elliptic arc is now reckoned from the end of the major instead of the minor axis, the zero point H however being always on the axis of Y.

The Locus of the Points of Contact F.—As the phase of the pen is $90° + \beta$ at the cusp C and $90° - \beta$ at the point of contact F, all that has been said about the former applies to the latter, provided we reverse the sign of β. The F points are thus seen to lie on the lobes of an equal equifoliated rosette with the equation $\rho = - n \sin n\theta$, these lobes being diametrically opposite to those of the cusp rosette. It has been omitted from Fig. 8.

The Tangent Method of Tracing the Rosette.—The mechanical method of tracing a rosette by having a pen move with a rectilinear simple harmonic motion of n cycles over a uniformly rotating disk, is mathematically a tangent method. If we imagine the two components of the pen's motion, the rectilinear over the line ERG and the rotary about A, to act successively instead of simultaneously, the pen is first at L in Fig. 7 in its rectilinear motion, in phases $90° + \beta$ and $90° - \beta$, and is then carried to C and F by the rotary motion on a circle with center A and radius $n \sin \beta$. The length of the tangent to this circle is then zero. At any other phase θ the pen in its harmonic motion is, say, at B', which may be anywhere, but which we may place on B in order not to congest the figure and which we accent in order to distinguish the two. While the rotary motion alone may be conceived first to carry the pen to D, the tangential harmonic motion then moves it along the tangent DP which is equal to $LB' = LR - B'R = \cos \beta - \sin \theta$, or



rather $\sin\theta - \cos\beta$, because the tangent DP in the case illustrated is really negative.

On Fig. 7 these tangents have been drawn to almost every 30° of the disk. Owing to the small numerical value of LE in the instance presented, the tangents to the *tangent* circle, as we may call it, are positive only between F and C, between phases $90° - \beta$ and $90° + \beta$, which are here 60° and 120°, and negative for all other phases. The points C and F on the curve in phases $90° \pm \beta$ are thus at minimum, and those in phases 90° and 270° at maximum distances from A. It is obvious, of course, that as the harmonic motion is n times as rapid as the circular, the tangent at any phase angle on the tangent circle runs to a point on the curve in n times that angle, so that 60° on the circle in Fig. 7 is joined to $3 \times 60° = 180°$ on the rosette, 330° on the circle to $3 \times 330° = 2 \times 360° + 270°$ on the curve, and so on.

The Equation of a Cuspidal Rosette.—The tangent DP in Fig. 7 ought to lead us to the equation of the rosette. The radius vector ρ or AP is seen to be such that

$$\overline{AP}^2 = \overline{AD}^2 + \overline{DP}^2$$

or

$$\rho^2 = a^2 n^2 \sin^2 \beta + a^2 (\sin (n\theta + \alpha) - \cos \beta)^2$$

in which a is the amplitude of the simple harmonic motion of the pen ER, which we may take as our unit, n, β, α, have their usual meaning and any assumed values, and θ is the position angle of D from $+X$, or the angle DAL. But it is the angle PAL that we need. Calling this ω and DAP δ, we have $\omega = \theta + \delta$, using the plus sign in general because when DP is negative, as it is in Fig. 7, it will reverse the sign of δ. Now δ is the angle whose tangent is DP/AD. Taking θ as the independent variable, we may find ρ and ω, but it does not seem possible to express the relation between ρ and ω in one equation without the help of θ.

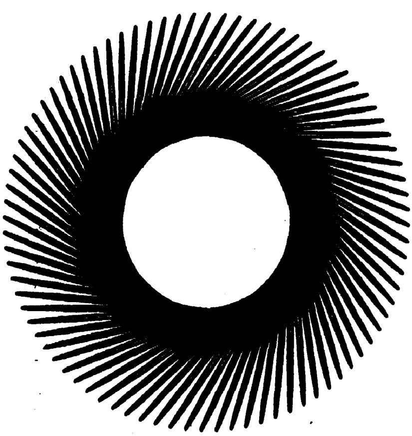
The Three Elements of a Cuspidal Rosette, n, α, β.—There are three elements that determine the shape and position of a cuspidal rosette, not to mention its size which depends upon the amplitude of the harmonic motion that we take as our unit. The first element is $n = p/q$, the ratio of the cycles of the pen p to those of the disk q. While n might be incommensurable, only simple ratios of integral numbers are here considered. The second element is α, the initial phase in its harmonic motion at which the pen is set down on the disk. This may have any value from 0° to 360°. In this paper α is taken as less than 90°, greater values having been treated in the cardioid article. The third element is β, the eccentric angle of the point B on the ellipse at which the pen is set down on the disk.

Variation in the Elements.—The nature of a rosette obviously depends upon $n = p/q$ and β, since p determines the number of lobes and cusps and q the number of its convolutions about the center A, while β, that is, $n \sin \beta$, modifies its shape by determining its distance from the center. There is then nothing left for α to do but to fix the position of the curve. For if the pen is first started in phase 0°, and then in phase α, the advance of the pen on the disk will be α/n, and the whole rosette will be shifted that angular amount forward in a clockwise direction. Hence the disk reading for $+X$ then becomes α/n instead of 0°, as it is in Fig. 8, where $\alpha = 0°$. In Fig. 7 therefore, where $\alpha = 20°$ and $n = 3$, the circle reading for $+X$ is $\alpha/n = 6° 40'$. The four points of the circle, 0°, 90°, 180°, 270° may thus be properly marked and inter-

mediate radii drawn for every 30° at pleasure. Then, as the circle reading for the axis of the first cusp lobe was found to be $180°/n$, its position angle from $+X$ then becomes $(180° - \alpha)/n$.

A variation in α alone shifts the center O of the ellipse along the Y axis to the distance $\sin \alpha$. It has no effect on the nature of the curve, as has been said, so that the rosette in Fig. 7, in which $n = 3$ and $\beta = 30°$ but $\alpha = 20°$, is exactly equal in every respect except as to the axes of X and Y to the second rosette on Fig. 8 in which also $n = 3$ and $\beta = 30°$ but $\alpha = 0°$. Even the position is the same in regard to the circle reading, because this is $(90° + \beta)/n$ for the cusp, independent of α, as we saw before.

A change of α, as has been said, affects merely the position of the rosette, so that it is drawn sooner or later than it was at the first value of α. While the curve is being drawn there is then no reason why we should not be able to take any instantaneous position of the pen as an initial position. As this initial position must be at the assumed starting point B at the eccentric angle β of the starting ellipse mentioned before whose center is the distance $\sin \alpha$ from A in the direction of B from R (Fig. 7), we may take this assumed point B on the ellipse, move it together with the ellipse along the rosette to a point in another phase, swing the ellipse about this second position of B until its center is at the distance of the sine of this new phase from A on a line parallel to the direction of the harmonic motion of the pen at the moment, and then mark the center on the paper. If we do this for all points of the rosette, we shall find the locus of the ellipse center O to be an equifoliated non-cuspidal rosette exactly like Fig. 5 when $n = 3$, with its lobe axes of unit magnitude lying on those of the contact F folium.



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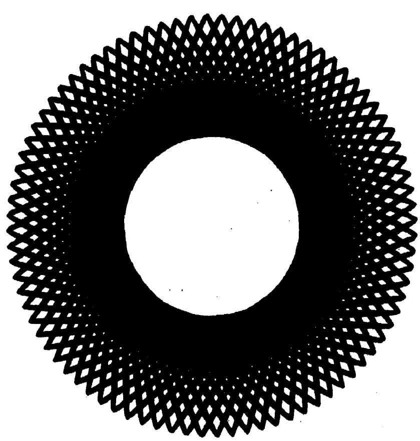

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The reason is that $AO = \sin \alpha$, and the angular revolution of the ellipse about B must be uniform for equal phase intervals, so that $d\theta$ is constant when $d\alpha$ is, as we always consider it to be. This O-folium may readily be drawn for any value of n, if we place the pen at A when in phase 0°. No illustrations of the O-folia are here given, since they would be for all values of n, as much like the original rosette when $\beta = 0°$ or 180° as Fig. 5 is like Fig. 1, that is to say, the O-folia would be non-cuspidal and of unit magnitude.

A variation in β alone does not affect the nature of the rosette when the old value of β is added to or subtracted from 180° and 360°. For this reason it is most convenient to use values of β less than 90°, and to express greater values in the way indicated. For $180° + \beta$ the new curve is symmetrical to the β rosette with respect to the center of the disk A. For $180° - \beta$ and $360° - \beta$ it is symmetrical to the β curve with regard to the $0° - 180°$ and $90° - 270°$ diameters respectively of the circle reading.

But it is the variations in n that teach us most about rosettes. Accordingly Figs. 9-25 show some typical cases. For Figs. 9-17 the initial phase α has been taken as 52° for the sake of comparison. The starting point B and the center O of the starting ellipse and the extremities of its axes have everywhere been marked. The dozen radiating dashes indicate every 30° of the circle, the 0° being marked by a cross. This last applies also to Fig. 2, in which however $\alpha = 90°$ and $\beta = 0°$. In Figs. 9, 10, 11, $n = 4/3$ and $\beta = 30°$, 90°, 180°, while in Figs. 12, 13, 14, $n = 3/4$ with the same values of β. In Figs. 15, 16, 17, $n = 1/2$ and $\beta =$ 30°, 60°, 180°. When the denominator q in n is large, the number of convolutions of the curve are uninterestingly numerous in proportion.

As α determines only the position of a curve, and this is generally of no consequence, the most convenient value of α to use is 90° when $\sin \alpha$ is a maximum and the pen is at one extremity of its harmonic path. Then when n is large and β small, the starting point is practically on the X axis. Thus Figs. 18, 19, 20 show the ratio $n = 12$ and $\beta = 0°$, 3°, 19°, respectively. In Figs. 21, 22, 23, $n = 24/5$ and $\beta = 0°$, 1° 4′, 5° 42′, and in Figs. 24 and 25 $n = 84$, $\beta = 0°$ 38′ and $\pm$ 0° 38′. Fig. 25 is in a sense merely the double of Fig. 24. It was drawn by first tracing Fig. 24 with the B point to the right of the center, and then by starting the pen an equal distance to the left. The cuspidal nature of some of the rosettes presented is not conspicuous, especially when β is very small, on account of the closeness and apparent superpositions of the paths of the pen when near the cusp. While other and more beautiful rosettes could have been drawn, had the restrictions as to cusps been waived, they would have been foreign to the present study.

Envelope Rosettes

One way of drawing a cardioid is to make a pen start in phase 90° from the center of the disk and move along a radius with simple harmonic motion, while the disk revolves with a uniform angular speed of the same period. If now, instead of a *simple* harmonic movement with the equation $\varrho = 1 - \cos\theta$, the amplitude being unity, we give the pen a *double* harmonic motion, and write the equation

$$\varrho = (1 - \cos\theta) + (1 - \cos m\theta),$$

in which m differs from unity by some small aliquot fraction; we shall then get a series of harmonic curves which have only one variable parameter, and which must therefore have a common envelope. The problem before us is to find this envelope.

The Inner Envelope a Cardioid.—The method of procedure, according to the textbooks, is to differentiate the above equation by regarding the variable parameter m as the only variable in it, and then to eliminate the parameter between these two equations. This will give us

$$\sin m\theta = 0,$$

and then

$$\pm 1 = \varrho - 2 + \cos\theta,$$

so that the equation of the envelope becomes

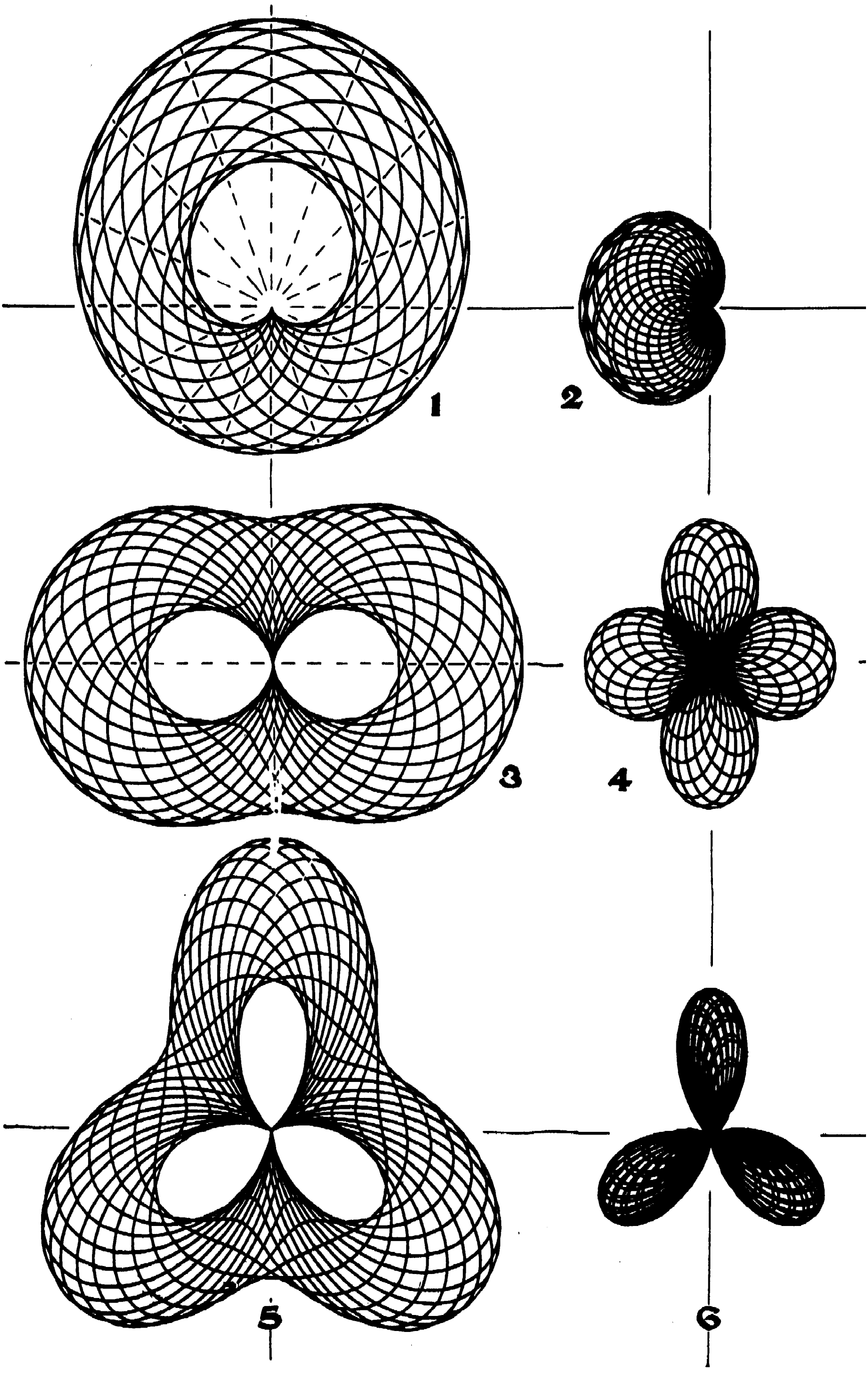
$$\varrho = (1 - \cos\theta) + 2$$

and

$$\varrho = 1 - \cos\theta.$$

There is therefore an inner envelope which is a cardioid, and an outer one in which the radius vector of this cardioid is increased by 2.

Fig. 1 has been drawn to represent these curves. This figure, like all those in this article, has the same position on the page that it had on the disk at the moment when it was completed and the drawing pen had returned to its initial position. If we can imagine the disk, which turns in a clockwise direction, to be now alone arrested, the pen will keep on moving up and down along the vertical line of the page through the cusp, that is, along what is generally denoted as the Y axis in figures, but which we may call here the mechanical axis of Y. The mathematical axis of $+Y$ which is used in the equations just given and which convention directs to run always upward, runs to the right in this Fig. 1, so that the figure must be turned 90° in an anticlockwise direction in order to have it oriented in the usual way. The reason for this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departure from the customary mathematical practice was that, by presenting the mechanical aspect of the figures, the changes that come over them when the initial phase or position of the pen or the rotation frequency of the disk is altered, may be seen to better advantage. The mathematical axes must therefore be rotated to suit each figure in particular. This will present no great difficulty.

The motion of the pen in Fig. 1 was the resultant of two simple harmonic movements both of the same amplitude, one with a unit period and the other with a period m, $^{15}\!/_{16}$ or $^{16}\!/_{15}$ as long, while the disk had a period of either component. In practice component A had a wheel with 32 cogs which made 15 revolutions while component B with 30 cogs made 16, the disk in the meantime with a 30- or 32-cog wheel making 16 or 15 turns. In Fig. 1 a 32-cog wheel with 15 revolutions was used on the disk. A radius (through the cusp) may be seen to cut the compound curve in 15 points. Had a 30-cog wheel with 16 revolutions been employed, there would have been 16 such intersections.

The pen was placed at the center of the disk (at the cusp) when both of its components were in phase 90°. When set in motion the pen started to draw a cardioid twice the size of the inner envelope, but this at once, although gradually, changed into a curve that became more and more curtate as the pen receded farther from the center at each revolution, until, in the middle of its compound period, when $\cos\theta + \cos m\theta$ was equal to zero, it momentarily drew the arc of a circle with the radius 2. After this the lobes of the curve repeated themselves in inverse order, while their axes kept on swinging in the same direction.

A study of Fig. 1 shows that the points of intersection of the lobes are arranged in radial lines at equal angular intervals, and that the points of tangency of the curve with the two envelopes are also spaced equiangularly.

The Outer Envelope a Cardioid.—There is a second way of drawing an envelope that is a cardioid. In the first case we placed the pen at the center of the disk when the phase of each of its two harmonic components was 90°. Now let us make the phase 0° at the center. The first component A, if used alone, will then trace the circle $\varrho = \sin\theta$, and the two together will trace $\varrho = \sin\theta + \sin m\theta$. Proceeding as before, we find the envelope

$$\pm 1 = \varrho - \sin\theta$$

or

$$\varrho = 1 + \sin\theta$$

and $\varrho = -1 + \sin\theta$ or $(1 + \sin\theta) - 2$, which are identical, or rather coincident, the first being traced as usual by the positive extremity of

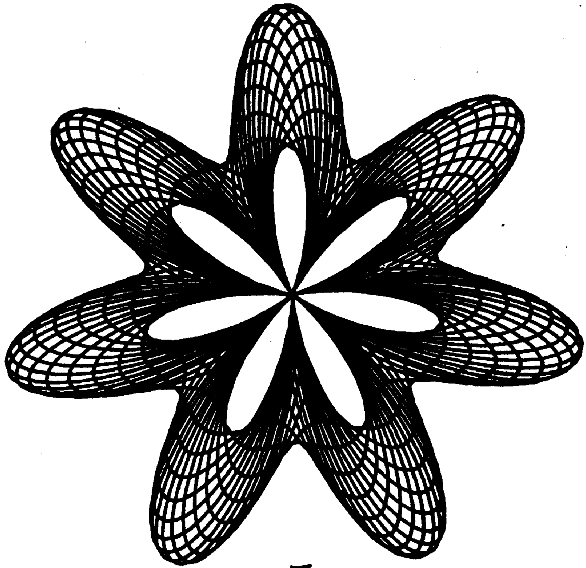
ρ, say by point P, and the other by a point P' at the constant distance of 2 from P in the negative direction of ρ. There is then practically only one envelope, which we may in the mechanical sense call an outer one. The mathematical axis of $+Y$ now runs to the left in Fig. 2. We may note that the equation of the second envelope of Fig. 2 has the constant -2 as opposed to $+2$ in the second one of Fig. 1.

Envelope Rosettes.—Generalizing the above results by using $n\theta$ in place of θ, we may apply the same principles to rosettes. Thus if we take $n=2$, there are two complete compound cycles of the pen to one of the disk, that is, the components A and B turn twice 15 and 16 times while the disk makes as before its usual 15 or 16 revolutions. We then have a rosette in the inner envelope of Fig. 3 and in the outer one of Fig. 4, using the latter expression in the mechanical sense. In the mathematical sense, however, there are two envelopes in Fig. 4, not coincident, but lying at right angles to one another. The direction of the mathematical axes has also undergone a change. Their position in these and subsequent figures may readily be deduced from the respective equations, and will for that reason no longer be referred to.

When $n=3$ we see that the usual inner envelope is a rosette in Fig. 5, and that the outer one in Fig. 6 is an equal one. Fig. 7 shows a septifolium as an inner envelope. The outer one was not drawn because it would have been almost totally black, as Fig. 6 leads us to suspect, on account of the great number of its close and overlapping lines.

From a mathematical point of view all these seven figures present envelopes that are rosettes. When the common phase of the two components A and B is made 90° at the center of the disk, as in Figs. 1, 3, 5, 7, we have two envelopes, an inner one which is a rosette, and an outer one in which the radius vector of the inner one is increased by 2. When the common phase is 0° at the center, as in Figs. 2, 4, 6, there are also two envelopes, both being equal rosettes. They are coincident when n is odd, but crossed equiangularly when n is even. From a mechanical standpoint, the first class of figures may be said to have rosettes as their inner envelopes and the second to have corresponding equal rosettes as their outer envelopes, the number of lobes being doubled in the latter case when n is even.

The Ratio of the Periods of the Components, or the Value of m.—In all the cases presented m was taken as $^{15}/_{16}$ or $^{16}/_{15}$, the harmonic component A making $15n$ revolutions and B $16n$, while the disk rotated 15 or 16 times. The number of revolutions of the disk, 15 or 16, must be $1/n$ that of one of the components. We may select either except



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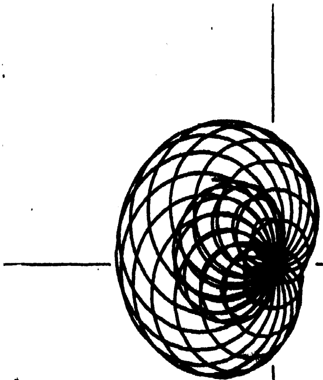
when n is a factor of the one used, for then, as soon as the pen has run through one complete compound cycle, it will begin to retrace the curve already drawn, so that the figure will present a disappointing appearance of incompleteness, since it will have only one nth as many lines as it ought to have. For this reason the disk had to make 15 turns for $n = 2$ and 16 for $n = 3$. For $n = 7$, 15 were made, but 16 would have done equally well. For $n = 5$ (not shown) they had to be 16.

The Starting Phases of the Components.—When the phases of the components A and B were 90° and the pen was set down at the center of the disk, the inner envelopes it traced were the rosettes shown in Figs. 1, 3, 5, 7. The identical figures, only turned at right angles, were drawn when the pen was started in phase 0° on the mechanical Y axis at the distance $+2$ from the center, that is, at the upper end of a lobe. The reason is that in a quarter of a turn of the disk one of the components A or B advances exactly one or n quarters of a period also and the other only one-fourth of $\frac{1}{15}$ or $\frac{1}{16}$ more or less. This difference is insensible in practice when $m - 1$ is very small. In like manner the identical "outer-envelope" rosettes, turned at right angles, resulted, Figs. 2, 4, 6, when the pen was started on the Y axis at the distance $+2$ in phase 90° instead of at the center in phase 0°. From this it follows that the pen may be started in any equal phase α of its components and set down on the mechanical Y axis at the distance $2 \sin \alpha$ from the center in the last case and $2 \cos \alpha$ in the first, in order to



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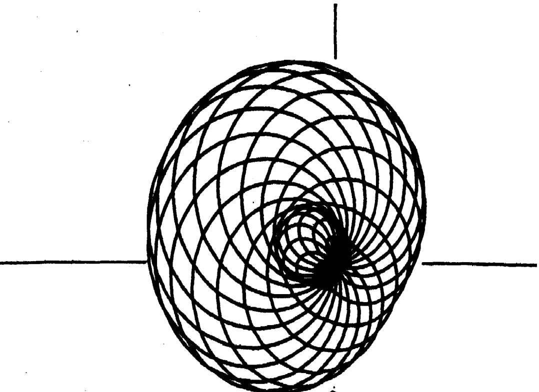

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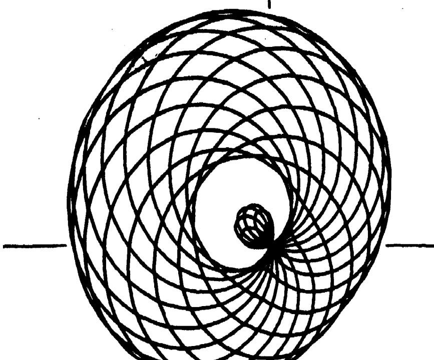

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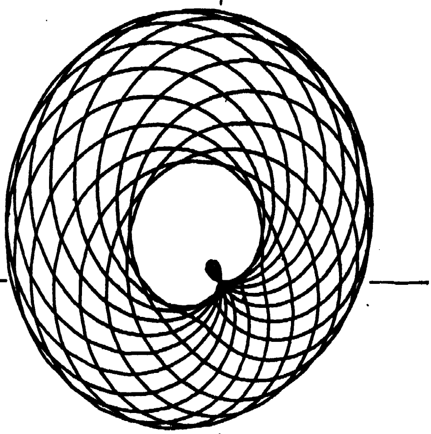

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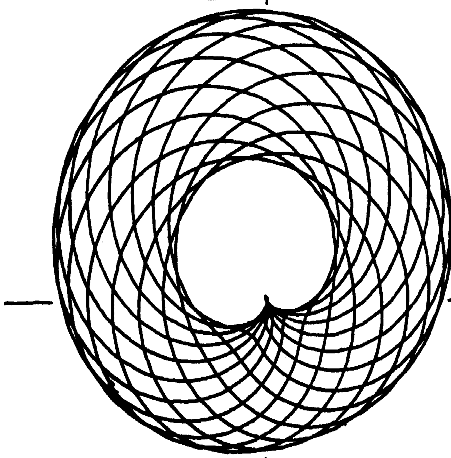

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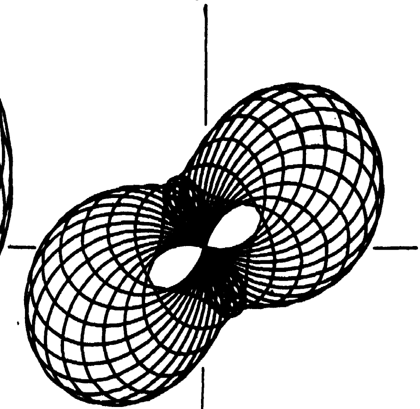


13

draw the same respective rosette, which will then be turned through the angle α on the disk. For this reason we might call the rosettes in Figs. 1, 3, 5, 7, "cosine" rosettes and those in Figs. 2, 4, 6, "sine" rosettes.

Transition Envelopes.—The principle just stated may be applied to show the transition from the (mechanical) outer- to the inner-envelope cardioid. Thus Figs. 8-12 are intermediate between Figs. 2 and 1. In all of these seven figures the starting point was at the center of the disk, but the phases were taken at 15° intervals from 0° to 90°. In Fig. 2 the phases of the pen at the start were 0°. In Fig. 8 the phases were 15°, in Fig. 9, 30°, in Fig. 10, 45°, in Fig. 11, 60°, in Fig. 12 75°, and finally in Fig. 1, 90°. The transition may thus be readily followed, and the axis of the envelope seen to swing round with uniform speed.

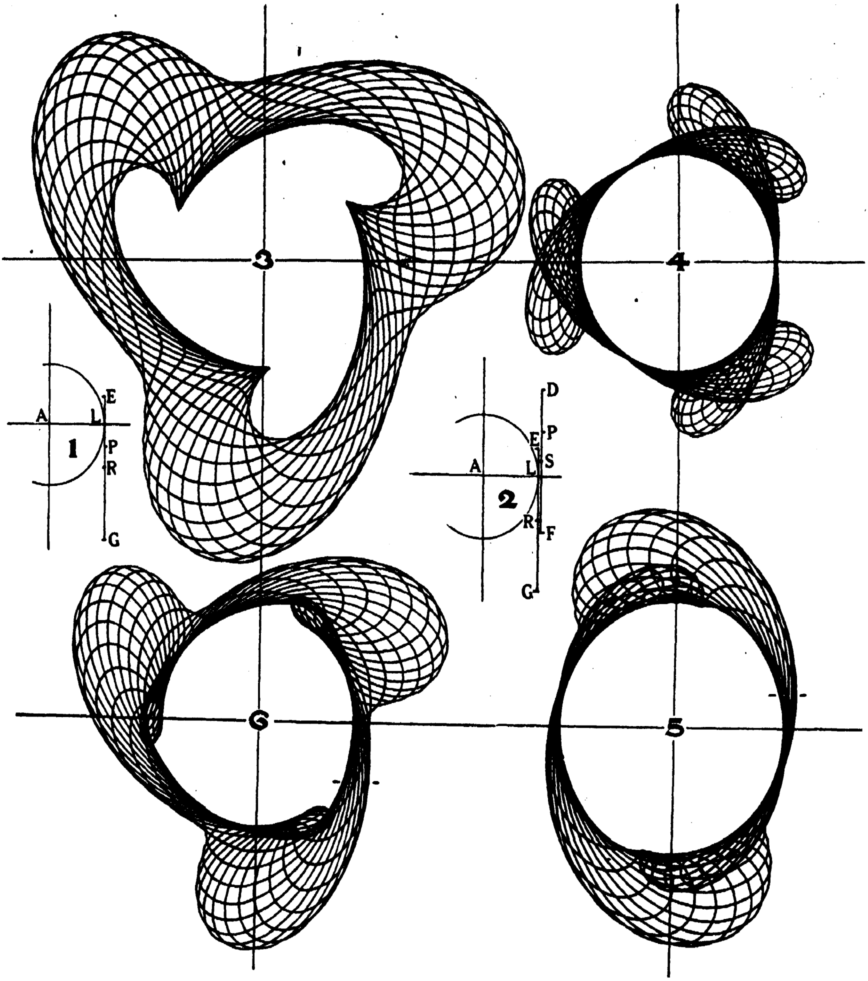
This identical series of seven transition envelopes might have been obtained by keeping the starting phases of the components at 90° and setting down the pen on the mechanical Y axis at the distance of twice the sines of 0°, 15°, $\cdots$ 90° from the center. In this case the axes of the envelopes would have remained stationary on the mechanical X axis. When the pen is set beyond the distance $2 \sin 90°$, the envelopes become curtate. Their inner faces will be the outer ones in Figs. 2, 8-12, 1, while their outer ones will tend to become more circular.

Unequal Starting Phases of the Components.—Instead of starting the pen with its components in equal phases, phase differences of any magnitude may be used. By studying the usual generation of Fig. 1 as given before, the initial position of the pen on the mechanical Y axis may so readily be deduced from the position it has there corresponding to the given phase difference, that numerical exemplifications are not necessary. The application to rosettes in general is also sufficiently obvious.

Finally Fig. 13 shows a transition envelope for an even value of n intermediate between Figs. 3 and 4.

Cuspidal Envelope Rosettes

A point P moves in the line segment EG, Fig. 1, with simple harmonic motion of p cycles, while this segment makes q revolutions about A with uniform angular speed. Moritz has exhaustively treated the case when the point A is in the line EG or in its prolongation. The writer has shown that when the point A is out of the line EG and the rosette drawn is cuspidal, AL, the distance of EG from A, must be $n \sin \beta$ (in which $n = p/q$) and LR, the distance of R, the mid-point, or point of zero phase, of EG, from its point of tangency L on the *tangent*



circle, must be cos β. The point P remains on an ellipse whose conjugate semi-axis is unity ($=ER=RG$) and is always parallel to EG, whose major semi-axis $=n$, and whose center is the sine PR of the phase α distant from A, β being the eccentric angle of P.

When the point P is given a double rectilinear harmonic motion with equal amplitudes but with periods in the ratio of m to 1, it may be conceived to move with simple harmonic motion of mn periods on the line segment DF, Fig. 2, while this line slides n times in a similar way along the line EG in one revolution of EG about A. For the sake of greater clearness these lines DF and EG are spaced a short distance apart in the figure. PS is then the sine of the phase of P on DF, while

RS is the sine of the phase of S, the mid-point of DF, on EG. Hence the distance of the tracing point P from the *tangent* circle measured along the tangent line $GFED$ is

$$LP = -LR + RS + SP = -\cos\beta + \sin n\theta + \sin mn\theta,$$

in which θ is the phase of the circular motion about A. In the previous paper the point A was in the line EG. In the present A will be out of this line. The discussion will, as before, be confined to envelopes that are cuspidal.

Two Envelopes.—As the points D and F are the limits of the excursions of P on this line segment DF, it is clear that these points themselves would trace the envelopes to all the lobes or loops or branches drawn by P, and that P must be on these envelopes when it is in phases 90° and 270°, respectively, on DF. The distance between the two envelopes measured in the line of motion of P is thus equal to 2, and the distance of P from them when in phase 0° or 180°, or multiples of them, must be -1 and $+1$.

The Starting Position of P for Cuspidal Envelopes. If D (or F) is to trace a cuspidal rosette, P must be set on a point of this rosette in phase 90° (or 270°) on DF, and then the phase of S on EG must also be set according to the nature of the curve. As the phase of S on EG, when $m-1$ is infinitesimal, may have any value corresponding to any phase of P on DF, we select the convenient value of $\theta = 0°$, when both will be zero together, to start P moving. Then P, R, S are coincident in Fig. 2. But as the distance of P from the D (or F) envelope will then be $+1$ or -1, this starting position of P, R, S, in phase 0° must therefore be the distance unity on one side or other of the cuspidal rosette, that is, its starting ordinate $y_0 = -\cos\beta - 1$ or $-\cos\beta + 1$, while x_0 is always $n \sin\beta$. From this it is evident that only one of the D and F envelopes can be cuspidal, except in a special case to be mentioned presently, when both may be so.

The Influence of β.—When y_0 or $-\cos\beta \pm 1$ is small numerically, the D and F envelopes will as a rule intersect. Although unequal, they are always symmetrical on account of the positions of E and G on opposite sides of L. They are equal only when $y_0 = 0$, that is, when $\beta = 0°$ or 180°, being coincident when n is odd, and symmetrically displaced when n is even. When y_0 or $-\cos\beta \pm 1$ is large numerically, the D and F envelopes cannot intersect, because E and G are then at very unequal distances from L and A. The inner one of the two will be cuspidal.

Illustrations.—Fig. 3 with $n = 3$, $\beta = 30°$, $x_0 = n\sin\beta = 1.5$, $y_0 = -\cos\beta - 1 = -1.866$, shows the D or inner envelope cuspidal,

while Fig. 4 with the same values of n, β, x_0, but with $y_0 = -\cos\beta + 1 = +0.134$, would seem at first sight to show two equal symmetrical and cuspidal envelopes. Only one however is in fact cuspidal and congruent to the inner envelope of Fig. 3. It is here the F envelope and is really smaller than the other or D envelope, because in this case $GL < EL$. The D envelope of Fig. 3 is in every way exactly equal to the F envelope of Fig. 4. When $\beta = 60°$ (but $n = 2$), as in Fig. 5, the inequality of the two envelopes is obvious, the smaller alone being cuspidal.

Transition Envelopes.—Fig. 6 shows a transition envelope for $n = 3$. When the tracing point P is started in phases 0° with a smaller numerical value of y_0 than $-\cos\beta - 1$ as in Fig. 3, the cusp throws a lobe or a shoot like a growing bud, while the outer envelope is contracted. While y_0 is increasing in the positive direction, the two envelopes may become apparently equal or nearly so, although neither can be cuspidal. The cuspidal stage will be reached again when $y_0 = -\cos\beta + 1$ as in Fig. 4. After that the growth just mentioned will be reversed, until the value of $\beta' = 180° - \beta$ will again give cuspidal envelopes symmetrical however to those of β. For still larger positive values of y_0, the envelopes will tend to become nearly equal and circular.

Cuspidal envelope rosettes are best drawn when n is a small integer. When $n = 1$ one or both of the envelopes are cardioids for all values of β.